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A Simple Linear Bayes' Credibility Insurance Model

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Abstract. In the minds of most statisticians there are (at least) two mutually exclusive approaches to data analysis. The "classical" or "frequentist" theory consist s of confidence intervals and hypothesis tests. And on the other hand, "Bayesian" statistics, a mode of inference is based on Bayes' Theorem. The goal of statistical inference is to extract and to report all available information about an unknown state of nature (parameter of interest) Θ.

Two important actuarial problems can be successfully solved by the Bayesian approach. The first is the general problem of model-based prediction and the second problem comes under the general name of credibility.

The goal of this paper is to show that the successful application of credibility theory would improve the ability of insurance companies to manage their financial systems with explicit risk components, also to show that even by adopting the Bayesian view, it was clear that the credibility solution is the best linear approximation to the Bayes solution of using the posterior mean.

1. Introduction

Credibility theory is a collection of ideas and techniques for the systematic adjustment of insurance premiums as claim experience is obtained. The need for these methods arose in situations where data from group of contracts were scanty and hence inadequate for providing reliable estimates of the risk premium [1, pp. 156-176]. In other words, the claim frequency rate for a class of insurance business may lie anywhere between 0 and ∞. An insurer with a large experience may have quite an accurate estimate of the rate. Nevertheless, it is an estimate and needs to be updated as further data come to hand. Also, a new insurer in the market will have little or no data of his own upon which to base the estimate of his claim frequency rate. The estimate will be rather uncertain (based perhaps on industry statistics and certain subjective judgements), and it is imperative that it be updated as soon as the insurer obtains reliable data. In either case, Bayesian methods may be employed in the updating [2].

2. The General Approach of Credibility

Let us assume that we need to estimate a parameter α. An $\hat{\alpha}_1$ can be obtained using only the individual data. It will, however, be unreliable if the individual data are sparse.

Alternatively, another estimate $\hat{\alpha}_c$ might be used, based on collateral data. Although the collateral data may be extensive, not all of them may relate to exactly the same risk. A trade off is needed between the two estimates $\hat{\alpha}_c$ and $\hat{\alpha}_1$.

The credibility theorist assigns a credibility factor Z to the individual data. The credibility factor Z is a number between 0 and 1, which is usually determined by both the individual and collateral experiences. It is close to 1 when the individual data are extensive and close to zero when they are sparse. A credibility estimate of the parameter of interest α is then given by

$$(1-Z)\hat{\alpha}_c + Z\hat{\alpha}_1 \qquad (1)$$

3. Credibility Theory's Role in Practice

In order to place credibility theory in a proper perspective, it is useful to review the steps in managing a financial system with explicit risk components [3, pp. 181-192]. The steps in this process are exhibited in Table 1.

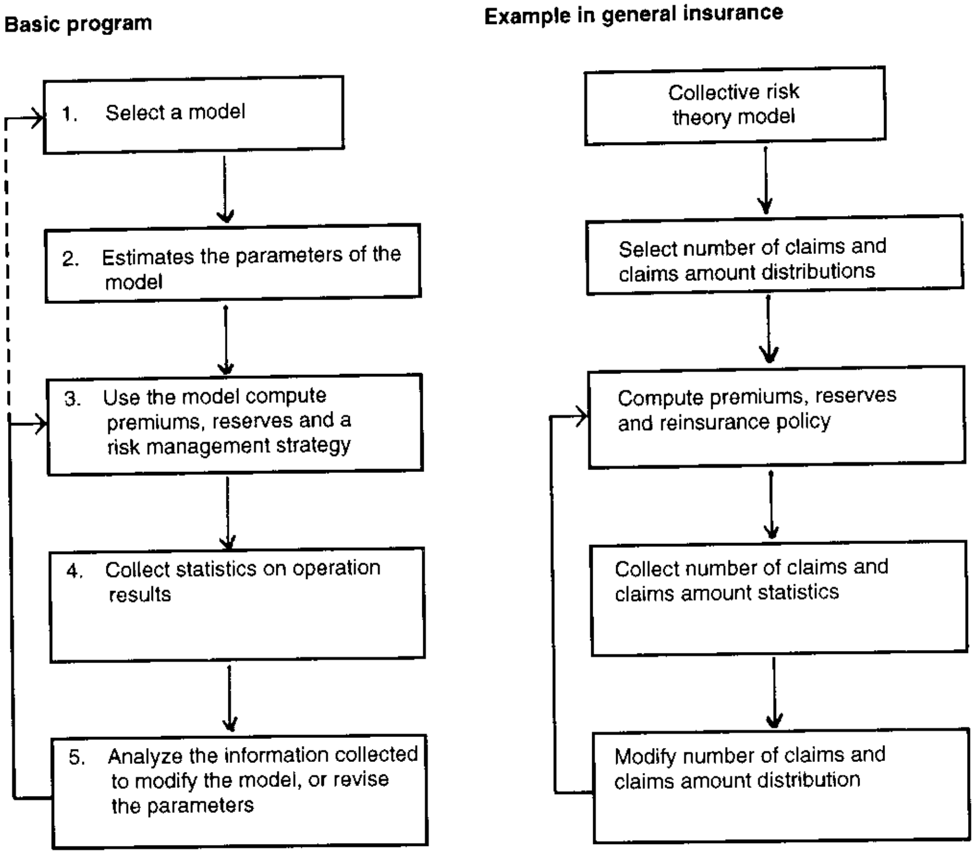
1. Selection of a risk model. This is by far the most important and most perplexing step. Keeping Theil's admonition in mind, the actuary must humbly remember that he can never capture reality in his models. On the other hand, only by the use of models can he plane.
2. Obtain initial estimates of the parameters of the model. It is inconceivable that an insurance system would be proposed without some prior feeling for the magnitude of the parameters.
3. Use the model and the initial parameters to compute premiums and reserves. Also, a second aspect of this step is to use the model to establish risk management policy (selection standards, reinsurance policy, etc.)
4. Collect information on the operation of the system.
5. Analyze the information obtained with the objective of possibly modifying the basic model. More likely the application of the new information will be in revising the estimates of the parameters of the model. From the revised estimates, adjustments in the price structure and the risk management strategy are possible. It is in this step that credibility procedures play a key role.

4. A Simple Model

Let there be k classes of policy holders and let there be t observations from each class. Let X_{ij} be the J^{th} observation from class i. Further assume that the probability distribution of X_{ij} depends on a (possibly vector) parameter Θ_i.

Also assume that given the parameters Θ_1, Θ_2, Θ_K the observations are independent. In practice the observations might be loss ratios from each class measured over t years or

Table 1. Steps in the management of a risk process



might be actual losses from t individuals in each class. In either case we want to learn about $X_{i,t+1}$, the next observation in class i.

The expected value is

$$E(X_{i,t+1} \setminus \Theta_i) = m(\Theta_i) \tag{2}$$

Our goal is to use the tk observations to obtain an estimate of this quantity.

4.1 Estimating the class mean

A logical first choice for an estimate of m (Θ_1) is

$$\overline{X}_i = \sum_{j=1}^{t} X_{ij} \setminus t \tag{3}$$

It is unbiased and enjoys the other standard properties of the sample mean [4]. With no additional knowledge it would be hard to do better.

Now suppose we know that the parameters $\Theta_1, \Theta_2, \ldots \Theta_k$ are independently distributed among the k classes according to some probability distribution that depends on a (possibly vector) parameter μ. There are two ways to think about this second level distribution [5, pp. 58-61]. One view is that there is a real, but unseen, process that has allocated these parameters to the classes. Learning about this allocation process is just another estimation problem [6]. The other is that this second level distribution represents our knowledge of the classes before collecting the observations. As long as the second level distribution is the same these two approaches will lead to the same result. Those who take the second approach, often call their analysis Bayesian, but this is correct only if their knowledge is formulated before collecting the data. In other words, we can say, the key point is that the classes are linked by the process that generated the class parameters.

To improve the sample mean as a point estimate of the next observation, we need a criterion for evaluating estimates. The measure of choice in the development of credibility theory has been mean squared error [7].

To make this precise, let X without a subscript stand for the entire collection of observations and let δ (X) be the estimate of m (Θ_i) based on these observations. The mean squared error of the estimate is

$$\text{m s e } [\delta(X)] = E_{\Theta}[E_{X/\Theta}\{[\delta(X) - m(\Theta_i)]^2\}] \tag{4}$$

where the expectation is taken over all possible values of $\Theta_1, \Theta_2, \ldots \Theta_k$ and of X.

For $\delta(X)=X_i$ the inner expectation is just the variance of the sample mean which is one over t times the variance of a single observation,

$$\text{Var } (X_{ij} / \Theta_i) = s(\Theta_i) \tag{5}$$

The mean squared error is then

$$E_{\Theta}[s\ (\Theta_i)] / t \tag{6}$$

which is a function of the second level parameter μ. This problem is a classical Bayes decision problem [8, pp. 15-40], and it is not difficult to see that the minimum mean squared error is achieved by taking.

$$\delta(X) = E_{\Theta/x}\left[m(\Theta_i) / X\right] \tag{7}$$

the posterior mean of the quantity of interest. It will be a function of the data and of the second level parameter μ. To evaluate this estimate we would not only need to know this parameter, but also the form of the distribution of X_{ij} and Θ_i.

Now let

$$\delta_c(X) = Z\ \overline{X}_i + (1-Z)m \tag{8}$$

Where

Z: is a credibility factor assigned to the individual data.

and m : is our best guess at the average value from class i prior to having collected any data and is a function of μ, i.e. $m = E_\Theta[m\Theta_i)]$

The inner expectation for the mean squared error is

$$\begin{aligned} &E_{x/\Theta}\Big\{Z^2\overline{X}_i^2 \ + \ 2Z\overline{X}_i\big[m - Zm - m(\Theta_i)\big] \\ &+(1-Z)^2 m^2 + m(\Theta_i)^2 - 2(1-Z)m\ \ m(\Theta_i)\ \Big\} \\ &= Z^2\Big[S(\Theta_i)/t + m(\Theta_i)^2\Big] + 2Zm(\Theta_i)\big[m - Zm - m(\Theta_i)\big] \\ &+(1-Z)^2 m^2 + m(\Theta_i)^2 - 2(1-Z)mm(\Theta_i) \\ &= Z^2 S(\Theta_i)/t + (1-Z)^2\big[m(\Theta_i) - m\big]^2 \end{aligned} \tag{9}$$

Now let

$$S = E_\Theta\big[S(\Theta_i)\big]$$

and $V = \mathrm{Var}_\Theta\big[m(\Theta_i)\big] = \Big\{\big[m(\Theta_i) - m\big]^2\Big\}$

$$= E_\Theta\Big[m(\Theta_i)^2\Big] - m^2$$

then the outer expectation in (4) yields

$$m\ \ s\ \ e\ \Big[{}^{\delta}c(X)\Big] = SZ^2/t + (1-Z)^2\ \ V \tag{10}$$

Taking the derivative with respect to Z and setting it equal to zero yields

$$Z = \frac{V}{V + S/t} \tag{11}$$

Inserting this back into (10) produces

$$m\ \ s\ \ e\ \Big[{}^{\delta}c(X)\Big] = \frac{s\ \ v\ /\ \ t}{v\ +\ \ s/t} \tag{12}$$

Since both v and s must be non-negative the m s e is less than s/t which was the m s e of the sample mean.

Conclusions

The above results represent the basics of modern credibility theory, and there are a number of conclusions that can be drawn from them:

1. There are several schools of thought about credibility, if one strives to identify the most comprehensive schools, two are easy to establish. The first is the class of methods that initially requires fixing the size of an insurance experience to which complete credibility will be attached for ratemaking purpose. Once this anchor is fixed, the problem becomes that of establishing, in some reasonable fashion, partial credibility weights which will be applied to smaller amount of insurance data. The second school of thought might be described as the Bayesian school. Within this school the expected claims for a group is viewed as being generated by a random process that may not be completely known. Prior and collateral information about expected claim is summarized in the form of a prior distribution. As the claims experience unfold Bayes' Theorem is used to find the process generating the parameters of the claim distribution.
2. In our simple model, when class to class relationships are added into the analysis, the sample mean is no longer the best estimate.
3. The amount of weight to put on the sample mean is a function of v, s and t. As t increases so does the weight. This reflects the increased reliability of the sample from the class in question. As s increases the weight decreases. This reflects the increased variability of the observations within a class, indicating the unreliability of the sample mean. As v increases the weight increases. This reflects the increased variability from one group mean to the next. As the other groups become more unlike the group in question their contribution should be reduced.
4. The credibility solution is the best linear approximation to the Bayes solution of using the posterior mean. The advantage of the linear approximation is that only the first two moments of the two distributions need be known. The Bayes solution requires that the complete distribution be known.

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نموذج مصداقية بايزي خطي وبسيط للتأمين

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ملخص البحث. هناك مشكلتان إكتواريتان مهمتان تم إيجاد حلول لهما بالاعتماد على المفهوم أو المدخل البيزي.

الأولى وهي المشكلة الأعم تتمثل في كيفية وضع أساس وبناء نموذج تقديري للخسائر.

والثانية هي بالطبع اختبار مدى مصداقية هذا النموذج التقديري مستقبليًا.

وهدف هذا البحث هو الإشارة إلى أن التطبيق الناجح لنظرية المصداقية سيزيد ويحسن من كفاءة المؤسسات التأمينية عند إدارة أنظمتها المالية، أيضًا يشير البحث ويبرهن أنه حتى في حالة تبني الأسلوب البيزي، فإنه من الواضح أن الحل المبني على أساس نظرية المصداقية هو أفضل الحلول الخطية وأقربها للحل البيزي.