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Estimation of Parameters and Truncation Point for the Truncated Gompertz Distribution

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Abstract. In this article we discuss the problem of estimating the parameters of the left and the right truncated Gompertz distribution when truncation points are unknown. Maximum likelihood estimators and estimators involving expected values of appropriate order statistics are derived. Asymptotic sampling errors of estimates are also given.

1. Introduction

Truncated distributions arise when sample selection and/or observation is not possible in some subregion of the sample space. This can occur as a consequence of actual elimination of part of the original data.

Charernkavanich and Cohen [1] discussed this problem for complete samples with a variety of estimation problems involving truncated normal, gamma, Weibull, lognormal and various other truncated distributions. Bain and Weeks [2] and Deemer and Votaw [3] gave the main results in the case of truncated exponential distribution using censored data, and Shalaby [4] has discussed the problem with a Weibul distribution for censored type II data.

The present study is concerned with a Gompertz distribution that is singly truncated at an unknown truncation point. The truncation point then becomes an additional parameter which must be estimated from sample data along with the primary distribution parameters. Both right and left truncation are considered.

2. The Gompertz Distribution

The Gompertz distribution is applicable as a model for surviving distributions which has an increasing hazard rate for the life of the creatures and systems. Prentice and Elshaarawi [5] have used this model in their studies and Elandt – Johnson and Johnson [6] have shown that the Gompertz distribution is widely used in actuarial works.

The probability density function (p.d.f.) of this distribution can be written as

$$f(x\ ;\ a,\ b) = b\ e^{ax}\ exp\left[\frac{b}{a}\left(1-e^{ax}\right)\right] \qquad (2-1)$$

for $0 < x < \infty$, $a > 0$, $b > 0$ (0 otherwise). The corresponding cumulative distribution function (c.d.f.) is

$$F(x\ ;\ a,\ b) = 1 - exp\left[\frac{b}{a}\left(1-e^{ax}\right)\right] \qquad (2-2)$$

and the moment generating functions is

$$E\left(e^{tx}\right) = \left(\frac{b}{a}\right)^{\frac{-t}{a}} e^{\frac{b}{a}}\ \Gamma\left(\frac{t}{a}+1\ ,\ \frac{b}{a}\right) \qquad (2-3)$$

where

$$\Gamma(u,\ \lambda) = \int_{\lambda}^{\infty} z^{u-1}\ e^{-z}\ dz = \Gamma(u) - \sum_{k=0}^{\infty} \frac{(-1)^k\ \lambda^{u+k}}{(u+k)\cdot k!} \qquad (2-4)$$

The mean μ_x, variance σ_x^2 and third standard moment α_3 of this distribution are:

$$\mu_x = a^{-1} J_1\ ,\quad \sigma_x^2 = a^{-2}\left(J_2 - J_1^2\right)\ , \alpha_3 = \frac{J_3 - 3J_2 J_1 + J_1^3}{\left[J_2 - J_1^2\right]^{3/2}} \qquad (2-5)$$

where

$$J_1 = e^{\frac{b}{a}}\left[\ell na - \ell nb - \gamma - \sum_{k-1}^{\infty} \frac{\left(-\frac{b}{a}\right)^k}{k \cdot k!}\right] \qquad (2-6)$$

$$J_2 = e^{\frac{b}{a}}\left[(\ell na - \ell nb - \gamma)^2 + \zeta(2) + 2\sum_{k=1}^{\infty} \frac{\left(-\frac{b}{a}\right)^k}{k^2 \cdot k!}\right] \qquad (2-7)$$

$$J_3 = e^{\frac{b}{a}}\left[(\ell na - \ell nb - \gamma)^3 - (2-\gamma)\ \zeta\ (3) + 3(\ell na - \ell na - \gamma)\ \zeta\ (2)\right]$$

$$-6\sum_{k=1}^{\infty} \frac{\left(-\frac{b}{a}\right)^k}{k^3 \cdot k!}\ ], \qquad (2-8)$$

$\gamma \doteq 0.5772$ is Euler's constant and $\zeta(z) = \sum_{k=1}^{\infty} k^{-z}$

is the zetta function defined by Gradshteyen and Ryzhik [7].

3. Truncation on the Left

When the distribution with p.d.f. (2.1) is truncated on the left at $x = \alpha$, the resulting truncated distribution becomes

$$f_L(x\ ;\alpha, a, b) = f(x; a, b) / \left[1 - F(\alpha; a, b)\right] \qquad (3.1)$$

for $\alpha < x < \infty$, $\alpha > 0$, $a > 0, b > 0$ (0 otherwise). And it follows from (2.1) and (2.2) that

$$f_L(x\ ;\alpha, a, b) = b\ e^{ax}\ \exp\left[\frac{b}{a}\left(e^{a\alpha} - e^{ax}\right)\right] \qquad x \geq \alpha \qquad (3.2)$$

zero elsewhere.

The likelihood function of a random sample of size n from a truncated distribution with p.d.f. (3.2) becomes

$$L(x_1 \cdots x_n; \alpha, a, b) = b^n\ e^{a\sum x_j}\ \exp\left[\frac{b}{a}\left(n\ e^{a\alpha} - \sum e^{ax_j}\right)\right] \qquad (3.3)$$

3.1. Maximum likelihood estimation (MLE)

On taking logarithms of (3.3) and differentiating, we have

$$\frac{\partial \ell nL}{\partial \alpha} = nbe^{a\alpha}$$

$$\frac{\partial \ell nL}{\partial a} = \sum_{j=1}^{n} x_j + \frac{b}{n}\left(n\alpha e^{a\alpha} - \sum_{j=1}^{n} x_i\, e^{ax_j} \right) - \frac{b}{a^2}\left(ne^{a\alpha} - \sum_{j=1}^{n} e^{ax_j} \right)$$

$$\frac{\partial \ell nL}{\partial b} = \frac{n}{b} + \frac{1}{a}\left(ne^{a\alpha} - \sum_{j=1}^{n} e^{ax_j} \right) \tag{3.4}$$

The function (3.3) attains its maximum value when α is as large as possible. Since α ⩽ x, we then have

$$\hat{\alpha} = y_1 \tag{3.5}$$

where y_1 is the smallest sample observation; i.e. the first order statistics. Estimators for a and b follow as solutions of equations

$$n\hat{a} + \hat{b}\left(n\ e^{\hat{a}y_1} - \sum_{j=1}^{n} e^{\hat{a}x_j} \right) = 0$$

$$\left(\sum_{j=1}^{n} x_j \right)\hat{a} + \hat{b}\left(ny_1 e^{\hat{a}y_1} - \sum_{j=1}^{n} x_j e^{\hat{a}x_j} \right) + n = 0 \tag{3.6}$$

The set of equations given in (3.6) do not have explicit solution for $\hat{a}$ and $\hat{b}$, however their solution can be obtained using numerical techniques.

3.2. Modified maximum likelihood estimation (MMLE)

It is recognized that the maximum likelihood estimator for α is biased. Since $\alpha < y_1$, the bias in small samples might be quite large. For this case and since the maximum likelihood estimators of α were likely to be unsatisfactory; then several modified estimators to truncation point are considered by replacing equation (3.5) with alternate functional relationships as was done by Shalaby [4] in connection with parameter estimation in Weibull distribution.

The p.d.f. of the first order statistic in a sample of size n from the truncated distribution (3.2) becomes

$$g_L(y_1;\alpha,a,b)=nbe^{ay_1}\ \exp\left[\tfrac{nb}{a}\left(e^{a\alpha}-e^{ay_1}\right)\right], \tag{3.7}$$

$\alpha < y_1 < \infty$, zero elsewhere.

The expected value of Y_1 becomes

$$E(Y_1)=\alpha+\tfrac{e^{\Delta}}{a}\left[-\gamma-In\Delta-\sum_{k=1}^{\infty}\frac{(-\Delta)^k}{k\cdot k!}\right] \tag{3.8}$$

Where $\Delta=\tfrac{nb}{a}\ e^{a\alpha}$

3.2.1 The first modification (MMLE-1)

Estimating equation (3.5) of the MLE is replaced by

$$E(Y_1)=y_1 \tag{3.9}$$

and the estimating equations for (MMLE-1) accordingly become:

$$\hat{\alpha}+\frac{e^{\hat{\Delta}}}{\hat{a}}\left[-\gamma-In\,\hat{\Delta}-\sum_{k=1}^{\infty}\frac{\left(-\hat{\Delta}\right)^k}{k\cdot k!}\right]=y_1 \tag{3.10}$$

plus the two equations of (3.6) with y_1 replaced by $\hat{a}$. These three equations must, of course, be solved numerically to obtain $\hat{a}$, $\hat{b}$ and $\hat{\alpha}$.

3.2.2 The second modification (MMLE-2)

Estimating equation (3.5) of the MLE is replaced by

$$\hat{\alpha}=E\ (y_1) \tag{3.11}$$

and the estimating equations for (MMLE-2) accordingly become:

$$\hat{\alpha} = \frac{1}{\hat{a}}\left[\ell n\,\hat{\Delta} - \ell n\left(n\,\hat{b}\right) - \hat{\gamma} - \sum_{k=1}^{\infty} \frac{\left(-\hat{\Delta}\right)^k}{k \cdot k!}\right] \tag{3.12}$$

plus the two equations of (3.6) with y_1 replaced by â. These three equations must be solved numerically to obtain â, $\hat{b}$ and $\hat{\alpha}$.

3.3 Sampling errors

The variance of the maximum likelihood estimator $\hat{\alpha} = y_1$ can be found as

$$V(Y_1) = E\left(Y_1^2\right) - E^2(y_1) \tag{3.13}$$

where

$$E\left(Y_1^2\right) = \alpha^2 + \frac{e^{\Delta}}{a^2}\left[(\gamma - a\alpha + \ell n\Delta)^2 + \zeta(2) - a^2\alpha^2 - 2a\alpha\sum_{k=1}^{\infty}\frac{(-\Delta)^k}{k \cdot k!} + 2\sum_{k=1}^{\infty}\frac{(-\Delta)^k}{k^2 \cdot k!}\right] \tag{3.14}$$

and $E(Y_1)$ is given by (3.8)

The asymptotic variance – covariance matrix of the MLE of â and $\hat{b}$ can be expressed as

$$V\left(\hat{a},\ \hat{b}\right) = \left[U_{ij}\right]^{-1}; \qquad i,\ j = 1,\ 2 \tag{3.15}$$

where:

$$U_{11} = -E\left(\frac{\partial^2 \ell nL}{\partial a^2}\right)$$

$$U_{12} = U_{21} = -E\left(\frac{\partial^2 \ell nL}{\partial a \partial b}\right)$$

$$U_{22} = -E\left(\frac{\partial^2 \ell nL}{\partial b^2}\right)$$

The exact expression for the expectations in the above matrix are difficult to obtain. However, in practice we would need the estimate for variance-covariance matrix which Cohen [8] proposed by using approximations

$$U_{11} = -\frac{\partial^2 \ell nL}{\partial a^2}\bigg|_{a=\hat{a},\ b=\hat{b}}$$

$$U_{12} = -\frac{\partial^2 \ell nL}{\partial a \partial b}\bigg|_{a=\hat{a},\ b=\hat{b}}$$

$$U_{22} = -\frac{\partial^2 \ell nL}{\partial b^2}\bigg|_{a=\hat{a},\ b=\hat{b}}$$

These are given below

$$U_{11} = \frac{2\hat{b}}{\hat{a}^2}\left(n\hat{\alpha}e^{\hat{a}\hat{\alpha}} - \sum_{j=1}^{n} x_j e^{\hat{a}x_j}\right) - \frac{\hat{b}}{\hat{a}}\left(n\,\hat{a}^{2}e^{\hat{a}\hat{\alpha}} - \sum_{j=1}^{n} x_j^2\, e^{\hat{a}x_j}\right)$$

$$-\frac{2\hat{b}}{\hat{a}^3}\left(ne^{\hat{a}\hat{\alpha}} - \sum_{j=1}^{n} e^{\hat{a}x_j}\right)$$

$$U_{12} = \frac{1}{\hat{a}^2}\left(ne^{\hat{a}\hat{\alpha}} - \sum_{j=1}^{n} e^{\hat{a}x_j}\right) - \frac{1}{\hat{a}}\left(n\hat{\alpha}e^{\hat{a}\hat{\alpha}} - \sum_{j=1}^{n} x_j e^{\hat{a}x_j}\right)$$

$$U_{22} = \frac{n}{\hat{a}^2} \tag{3.16}$$

On inverting the matrix (3.15), it follows that

$$V\left(\hat{a}\right) = U_{22} / \left(U_{11}\ U_{22} - U_{12}^2\right)$$

$$V\left(\hat{b}\right)=U_{11}/\left(U_{11}\ U_{22}-U_{12}^{2}\right)$$

$$Cov\left(\hat{a},\hat{b}\right)=-\ U_{12}/\left(U_{11}\ U_{22}-U_{12}^{2}\right) \quad (3.17)$$

4. Truncation on the Right

For truncation on the right at x = α, the p.d.f. of the resulting truncated distribution becomes

$$f_R(x;\alpha,a,b)=f(x;a,b)/F(\alpha;a,b), \qquad 0<x<\alpha$$

$$=0,\ \text{elsewhere} \quad (4.1)$$

on substituting (2.1) and (2.2) into this equation, we have

$$f_R(x;\alpha,a,b)=b\left[1-e^{-c}\right]^{-1}\ e^{ax}\ \exp\left[\tfrac{b}{a}\left(1-e^{ax}\right)\right], \quad 0<x<\alpha$$

$$=0,\ \text{elsewhere} \quad (4.2)$$

where : $c=\frac{b}{a}\left(e^{a\alpha}-1\right)$

The likelihood function of a random sample of size n from a truncated distribution with p.d.f. (4.2) becomes

$$L\left(x_1\cdots x_n;\alpha,a,b\right)=b^n\left[1-e^{-c}\right]^{-n}\ e^{a\Sigma x_j}\ \exp\left[\tfrac{b}{a}\left(n-\Sigma e^{ax_j}\right)\right] \quad (4.3)$$

4.1 Maximum likelihood estimation (MLE)

On taking logarithms of (4.3) and differentiating, we have

$$\frac{\partial \ell nL}{\partial \alpha}=-nbe^{a\alpha}\left(e^{c}-1\right)^{-1}$$

$$\frac{\partial \ell nL}{\partial a}=\sum_{j=1}^{n}x_j-\tfrac{b}{a^2}\left(n-\sum_{j=1}^{n}e^{ax_j}\right)-\tfrac{b}{a}\sum_{j=1}^{n}x_j\,e^{ax_j}-\tfrac{nb}{a^2}\left(e^{c}-1\right)^{-1}\left(1+a\alpha e^{aa}-e^{aa}\right)$$

$$\frac{\partial \ell n L}{\partial b} = \frac{n}{b} + \frac{1}{a}\left(n - \sum_{j=1}^{n} e^{a x_j}\right) + \frac{n}{a}\left(e^{c} - 1\right)^{-1}\left(1 - e^{a\alpha}\right) \tag{4.4}$$

The function (4.3) attains its maximum value when α is as small as possible. Since $\alpha \geqslant x$, we then have

$$\hat{\alpha} = y_n \tag{4.5}$$

where y_n is the largest sample observation; i.e. the n-th order statistic in a random sample of size n. Estimators for a and b follow as solutions of equations

$$n\hat{a}\left[1 - \hat{c}\left(e^{\hat{c}} - 1\right)^{-1}\right] + \hat{b}\left(n - \sum_{j=1}^{n} e^{\hat{a} x_j}\right) = 0$$

$$\hat{a}\sum_{j=1}^{n} x_j - \hat{b}\left[\sum_{j=1}^{n} x_j \; e^{\hat{a} x_j} + n y_n e^{\hat{a} y_n}\left(e^{\hat{c}} - 1\right)^{-1}\right] + n = 0 \tag{4.6}$$

where: $\hat{c} = \frac{\hat{b}}{\hat{a}}\left(e^{\hat{a} y_n} - 1\right)$

The solution of the set of equations given in (4.6) can be obtained using iterative techniques for solving a pair of simultaneous equations in two unknowns.

4.2 Modified maximum likelihood estimation (MMLE)

In this case and since the maximum likelihood estimator of α were likely to be unsatisfactory; then several modified estimators to truncation point are considered by replacing equation (4.5) with alternate functional relationships as was done by Shalaby [4] in connection with parameter estimation in Weibull distribution.

The p.d.f. of the n-th order statistic in random sample of size n from (4.2) is

$$g_R\left(y_n;\alpha,a,b\right) = nb\left[1 - e^{-c}\right]^{-n} \; e^{a y_n} \exp\left[\frac{b}{a}\left(1 - e^{a y_n}\right)\right]\left\{1 - \exp\left(1 - e^{a y_n}\right)\right\}^{n-1}$$

$$0 < y_n < \alpha, \qquad \textit{zero elsewhere} \tag{4.7}$$

The expected value of Y_n follows as

$$E(Y_n)=\frac{\left(1-e^{-c}\right)^{-n}}{a}\sum_{k=1}^{n}(-1)^k\binom{n}{k}\left\{\alpha a e^{-kc}-e^{\frac{kb}{a}}\left[\alpha a+\sum_{j=1}^{\infty}\frac{\left(-\frac{kb}{a}\right)^j}{j\cdot j!}\left(e^{a\alpha j}-1\right)\right]\right\}. \quad (4.8)$$

4.2.1 The first modification (MMLE-1)

Estimating equation (4.5) of the MLE is replaced by

$$E(Y_n)=y_n \quad (4.9)$$

and the estimating equation for (MMLE-1) accordingly become:

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}\left\{\hat{\alpha}\hat{a}e^{-k\hat{c}}-e^{\frac{k\hat{b}}{\hat{a}}}\left[\hat{\alpha}\hat{a}+\sum_{j=1}^{\infty}\frac{\left(-\frac{k\hat{b}}{\hat{a}}\right)^j}{j\cdot j!}\left(e^{\hat{a}\hat{\alpha}j}-1\right)\right]\right\}=\hat{a}\left(1-e^{-\hat{c}}\right)^n y_n \quad (4.10)$$

plus the two equations of (4.6) with y_n replaced by $\hat{\alpha}$. These three equations must, of course, be solved numerically to obtain $\hat{a}$, $\hat{b}$ and $\hat{\alpha}$.

4.2.2 The second modification (MMLE-2)

Estimating equation (4.5) for the MLE is replaced by

$$\hat{\alpha}=E(Y_n) \quad (4.11)$$

and the estimating equations for (MMLE-2) accordingly become:

$$\hat{\alpha}\hat{a}\left(1-e^{-\hat{c}}\right)^n=\sum_{k=1}^{n}(-1)^k\binom{n}{k}\left\{\hat{\alpha}\hat{a}e^{-k\hat{c}}-e^{\frac{k\hat{b}}{\hat{a}}}\left[\hat{\alpha}\hat{a}+\sum_{j=1}^{\infty}\frac{\left(-\frac{k\hat{b}}{\hat{a}}\right)^j}{j\cdot j!}\left(e^{\hat{a}\hat{\alpha}j}-1\right)\right]\right\}. \quad (4.12)$$

plus the two equations of (4.6) with y_n replaced by $\hat{\alpha}$. These three equations are solved numerically to obtain $\hat{a}$, $\hat{b}$ and $\hat{\alpha}$.

4.3 Sampling errors

The variance of the maximum likelihood estimator $\hat{a} = y_n$ can be found as

$$V(Y_n) = E\left(Y_n^2\right) - E^2(Y_n) \tag{4.13}$$

where

$$E\left(Y_n^2\right) = \frac{\left[1-e^{-c}\right]^{-n}}{a} \sum_{k=1}^{n} (-1)^k \binom{n}{k} \left\{ \alpha^2 a^2 e^{-kc} - e^{\frac{kb}{a}} \left[\alpha^2 a^2 - 2\alpha a + 2\sum_{j=1}^{n} \frac{\left(-\frac{kb}{a}\right)^j}{j^2 \cdot j!} \left(e^{\alpha a j} - 1\right) \right] \right\}. \tag{4.14}$$

and $E(y_n)$ is given by (4.8)

The asymptotic variance-covariance matrix of the MLE of $\hat{a}$ and $\hat{b}$ is essentially the same as (3.15). In this case, however, the elements U_{ij} are

$$U_{11} = \frac{\hat{b}}{\hat{a}} \sum_{j=1}^{n} x_j^2 \; e^{\hat{a} x_j} - \frac{2\hat{b}}{\hat{a}^2} \sum_{j=1}^{n} x_j \; e^{\hat{a} x_j} - \frac{2\hat{b}}{\hat{a}^3} \left\{ n - \sum_{j=1}^{n} e^{\hat{a} x_j} - \left(e^{\hat{c}} - 1\right)^{-1} \left\{ \left(\hat{a}^2 \hat{\alpha}^2 - 2\hat{a}\hat{\alpha} + 2 \right) e^{\hat{a}\hat{\alpha}} - 2 \right\} \right\}$$

$$- \frac{n \hat{b}^2}{\hat{a}^4} \; e^{\hat{c}} \left(e^{\hat{c}} - 1\right)^{-2} \left(1 + \hat{a}\hat{\alpha} e^{\hat{a}\hat{\alpha}} - e^{\hat{a}\hat{\alpha}}\right)^2$$

$$U_{12} = \frac{1}{\hat{a}} \sum_{j=1}^{n} x_j \; e^{\hat{a} x_j} + \frac{1}{\hat{a}^2} \left\{ n - \sum_{j=1}^{n} e^{\hat{a} x_j} + n\left(e^{\hat{c}} - 1\right)^{-1} \left(1 + \hat{a}\hat{\alpha} e^{\hat{a}\hat{\alpha}} - e^{\hat{a}\hat{\alpha}}\right) \right\}$$

$$- \frac{n\hat{b}}{\hat{a}^3} \; e^{\hat{c}} \left(e^{\hat{c}} - 1\right)^{-2} \left(e^{\hat{a}\hat{\alpha}} - 1\right) \left(1 + \hat{a}\hat{\alpha} e^{\hat{a}\hat{\alpha}} - e^{\hat{a}\hat{\alpha}}\right)$$

$$U_{22} = \frac{n}{\hat{b}^2} - \frac{n}{\hat{a}^2} \left(1 - e^{\hat{a}\hat{\alpha}}\right)^2 \; e^{\hat{c}} \left(e^{\hat{c}} - 1\right)^{-2} \tag{4.15}$$

5. Illustrative examples

The practical application of estimators resulting in this work are illustrated with simulated data from the left and the right truncated Gompertz distribution. The numerical techniques and iteration processes of IMSL (1980) routines will be used in the following two examples.

5.1 Example (1), left truncation

In this example we have a random sample consisting of 30 observations from a left truncated Gompertz population in which $\alpha = 2$, $a = 1$ and $b = 0.01$. The individual observations are listed below:

3.46	4.87	4.76	2.83	4.99	4.44
3.30	4.24	5.18	4.31	5.30	2.23
3.16	3.59	5.65	3.82	3.83	4.08
2.38	2.57	4.00	4.56	3.80	3.30
5.96	4.12	4.02	4.15	2.81	4.55

For this sample $n = 30$; $\bar{X} = 3.492$, $S = 0.965$, $a_3 = 0.069$; y_1 2.23.

MLE, MMLE-1 and MMLE-2 calculations are summarized in Table 1. Approximate variance and covariance for these estimators are given also in Table 1.

Table 1. Estimates from left truncated Gompertz distribution

Method of estimation	MLE	MMLE-1	MMLE-2
α	2.23	2.155	2.161
a	0.846	0.871	0.869
b	0.025	0.023	0.023
var (α)	0.02323	0.02829	0.02792
var (a)	0.02418	0.01190	0.01225
var (b)	0.00025	0.0009	0.00009
cov (a,b)	−0.00236	−0.00092	−0.00096

5.2 Example (2), right truncation

In this example we have a random sample consisting of 20 observation from a right truncated Gompertz population in which $\alpha = 5$, $a = 1$ and $b = 0.01$.

Individual observations are tabulated below:

4.54	4.62	4.68	4.92	
1.00	3.28	4.00	4.48	4.48
4.88	4.11	3.43	2.77	
1.79	4.20	4.18	4.80	
3.93	4.33	3.67	4.08	

For this sample $n = 20$; $\bar{X} = 3.884$, $S = 0.998$, $a_3 = -1.533$, y_n 4.92.

MLE, MIMLE-1 and MMLE-2 calculations are summarized in Table 2. Approximate variance and covariance for these estimators are given also in Table 2.

Table 2. Estimates from right truncated Gompertz distribution.

Method of estimation	**MLE**	**MMLE-1**	**MMLE-2**
α	4.92	4.961	4.975
a	0.632	0.961	0.997
b	0.021	0.008	0.007
var (α)	0.01103	0.00906	0.00918
var (a)	0.02728	0.04131	0.05033
var (b)	0.00001	$0.2\text{x}10^{-5}$	$0.1\text{x}10^{-5}$
cov (a,b)	−0.00009	−0.00005	−0.00005

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تقدير المعالم ونقطة البتر لتوزيع جومبيرتز المبتـور

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(قدم للنشر في ٥/١/١٤١٢هـ وقبل للنشر في ١٩/١٠/١٤١٢هـ)

ملخص البحث. يتناول البحث مشكلة تقدير معالم توزيع جومبيرتز المبتور من اليمين ومن اليسار عندما تكـون نقـطة البتر غير معلومة حيث يمكن اعتبارها عندئذ معلمة إضافية يجب تقديرها. ولقد تناولت الدراسة ثلاث طرق للتقدير وهي طريقة الإمكان الأعظم وطريقتان معدّلتان للإمكان الأعظم وتم اختبار النتائج على بيانات مأخوذة من التوزيع المبتور باستخدام الحاسب الآلي.