J. King Saud Univ., Vol. 4, *Admin. Sci.* (1), pp. 53-64 (A.H. 1412/1992)

One and Two Sided Shortest Confidence Interval for the Weibull Shape Parameter Using the Two Quotient Failure Times

O.A. Shalaby

Associate Prof., Department of Quantitative Methods, College of Administrative Sciences, King Saud University, Riyadh, Saudi Arabia

(Received 20/2/1411; Accepted for Publication 28/12/1411)

Keywords: Weibull distribution; order statistics; shortest confidence intervals.

Abstract. This paper is concerned with estimation of the shape parameter of the Weibull distribution from small samples using one and two sided shortest confidence intervals based on the two quotient failure times. This approach is a generalization of that of Ashour and Jones [1] they obtained a shortest one sided confidence interval for the shape parameter based on two adjacent failure times.

1. Introduction

The Weibull distribution is an important model in the study of life testing experiments. First suppose we have $t_{(1)}, t_{(2)}, \ldots, t_{(n)}$; then ordered life times from the two parameter Weibull random variable T. The pdf and cdf of T are defined by

$$f(t) = \frac{b}{\theta} t^{b-1} \exp(-t^b / \theta); \quad \theta, b > 0, t > 0,$$

and

$$F(t) = 1 - \exp(-t^b / \theta),$$

where, in this work θ be the scale parameter and b is the unknown shape parameter will be estimated. A random variable $W = \ln T$ follows the extreme value distribution with cdf

$$F(w) = 1 - \exp[-\exp\{(w-\beta) / \delta\}], -\infty < W < \infty, \tag{1.1}$$

where $\delta = \frac{1}{b}$ and $\beta = \ln \theta$, are respectively the scale and location parameters of the above extreme value distribution.

Inference on the unkown parameters b or $\delta = 1/b$ may be carried out using few order statistics $t_{(i)}$ and $t_{(j)}$ $(1 \leq i < j \leq n)$. Jaech [2] considered point and interval estimation of b based on the ratio of $t_{(1)}$ and $t_{(2)}$, Bain [3] obtained an unbiased estimator for b when there are only two failures, Mann [4] considered the estimation of the Weibull shape parameter based on few order statistics, Murthy and Swartz [5] used the distribution of the logarithm of the ratio of any two order statistics for the unbiased estimator of b and testing hypotheses about b and Hassainian [6] considered the best linear unbiased estimator of $\delta = 1/b$ based on two or three order statistics. Procedures for point and interval estimation of b and δ based on two or more failures per lot are presented in Bain [7] and Engelhardt and Bain [8]. Ashour and Jones [1] obtained the exact one sided shortest confidence interval for b based on the ratio of $t_{(i)} / t_{(i-1)}$ and Salem [9] pointed out the one sided confidence interval for b based on the ratio $t_{(j)} / t_{(i)}$, $1 < i < j < n$.

In this paper we will use a function of the ratio $t_{(i)} / t_{(j)}$, to construct one and two sided shortest confidence interval for the shape parameter b.

It is required in this work to define the distribution of the ratio $t_{(i)} / t_{(j)}$, $i<j$ from the joint probability density function of $t_{(i)}$ and $t_{(j)}$. This joint density of $t_{(i)}$ and $t_{(j)}$ after the transformation $x = t_{(i)}$ and $y = t_{(j)}$ is

$$f(x,y) = k[F(x)]^{i-1} [F(y) - F(x)]^{j-i-1} [1 - F(y)]^{n-j} f(x) f(y); \; x<y$$

where

$$k = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}$$

Using F(t) and f(t) for the Weibull distribution and expanding by binomial theorem, the quantities that are raised to (i-1) and (j-i-1) power, we obtain the joint density of x and y

$$f(x,y) = \frac{kb^2}{\theta^2} x^{b-1} y^{b-1} \sum_{r=0}^{i-1} \sum_{s=0}^{j-i-1} c_{rs} \exp\{-\frac{1}{\theta}(Ax^b + ay^b)\}, \; x<y$$

where

$$C_{rs} = (-1)^{r+s} \binom{i-1}{r} \binom{j-i-1}{s}, \; a = n - j + s + 1 \; and$$

$$A = j - i + r - s$$

Using the transformation Z = x/y, we find, from Mood, Graybill and Boes [10] the pdf of Z as

$$f(z) = \int_{o}^{\infty} |y|\, f(zy, y)\, dy$$

$$= \frac{kb^2}{\theta^2} z^{b-1} \sum_{r=0}^{i-1} \sum_{s=0}^{j-i-1} C_{rs} \int_{0}^{\infty} y^{2b-1} \exp\left\{-\frac{y^b}{\theta}\left(a + Az^b\right)\right\} dy$$

$$= kbz^{b-1} \sum_{r=0}^{i-1} \sum_{s=0}^{j-i-1} C_{rs} A^{-2} \left(\rho + z^b\right)^{-2} \qquad (1.2)$$

where

$$\rho = \frac{a}{A}, \quad b > 1 \quad and \quad 0 < z < 1$$

When b = 1, (1.2) reduced to the pdf of the quotient of the *ith* and *jth* order statistics from the one parameter exponential distribution.

The pdf (1.2) was obtained also by Malik and Trudel [11] but they used the Mellin transform technique.

2. The Shortest Confidence Interval

Let U = – bln Z, the pdf of U is obtained from (1.3) as

$$f(u) = k \sum_{r=0}^{i-1} \sum_{s=0}^{j-i-1} C_{rs} A^{-2} \exp(-u) \left\{\rho + \exp(-u)\right\}^{-2}, \qquad u > 0 \quad (2.1)$$

The m-th non-central moment of U is given by

$$\mu'_m = k \sum_{r=0}^{i-1} \sum_{s=0}^{j-i-1} C_{rs} A^{-2} \rho^{-1} \sum_{\lambda=0}^{\infty} \frac{(-1)^{\lambda}\, \Gamma(m+1)}{\rho^{\lambda+1} (\lambda+1)^m}, \qquad m = 1, 2, 3, \ldots$$

Especially for m = 1,2, we have

$$E(u) = k \sum_{r=0}^{i-1} \sum_{s=0}^{j-i-1} C_{rs} A^{-2} \rho^{-1} \sum_{\lambda=0}^{\infty} \frac{(-1)^{\lambda}}{\rho^{\lambda+1}(\lambda+1)} =$$

$$k \sum_{r=0}^{i-1} \sum_{s=0}^{j-i-1} C_{rs} A^{-2} \rho^{-1} \ln\left(\frac{\rho+1}{\rho}\right)$$

and

$$E(u^2) = 2k \sum_{r=0}^{i-1} \sum_{s=0}^{j-i-1} C_{rs} A^{-2} \rho^{-1} \sum_{\lambda=0}^{\infty} \frac{(-1)^{\lambda}}{\rho^{\lambda+1}(\lambda+1)^{\lambda}}$$

$$= k \sum_{r=0}^{i-1} \sum_{s=0}^{j-i-1} C_{rs} A^{-2} \left[\ln^2\left(\frac{\rho+1}{\rho}\right) + 2 \sum_{\lambda=1}^{\infty} \frac{1}{\lambda^2 (\rho+1)^2}\right]$$

The probability statement $p_r\{u_0<u<u_1\} = 1-\alpha$ is converted to $P_r\{Q_1<b<Q_2\} = 1-\alpha$, where Q_1 and Q_2 are the lower and upper limits of b with confidence level $(1-\alpha)$ and $0<\alpha<1$.

The technique of finding a two sided shortest confidence interval reduced to solving the following two equations simultaneously, (see Guenther [12] and Akhlaghi and Parsian [13]).

$$f_U(u_0) = f_U(u_1)$$

and

$$\int_{u_0}^{u_1} f(u)\, du = 1-\alpha$$

where f(.) defined by (2.1)

Now the two sided shortest confidence interval for b is $\{u_0/-\ln z, u_1/-\ln z\}$ where u_0 and u_1 are obtained as solution of

$$\sum_{r=0}^{i-1} \sum_{s=0}^{j-i-1} C_{rs} A^{-2}\left[\frac{\exp(-u_0)}{\{\rho+\exp(-u_0)\}^2} - \frac{\exp(-u_1)}{\{\rho+\exp(-u_1)\}^2}\right] = 0, \qquad (2\cdot 2)$$

and

$$\left[\exp(-u_0) - \exp(-u_1)\right] \sum_{r=0}^{i-1} \sum_{s=0}^{j-i-1} C_{rs} A^{-2}$$

$$\left[\{\rho + \exp(-u_0)\}\{\rho + \exp(-u_1)\}\right]^{-1} = \frac{1-\alpha}{k}$$

Note that, Jaech [3] obtained an equal tail confidence interval for b based on the ratio $t_{(2)}/t_{(1)}$.

The point estimator of b is obtained by Murthy and Swartz [5] as

$$\hat{b}\ uB\ (i, j, n) \tag{2.3}$$

where

$$B(i, j, n) = b / E(u)$$

Note that

$$E(\hat{b}) = b,$$

and

$$var(\hat{b}) = B^2(i, j, n)\ var(u) = b^2\left[\frac{E(u^2)}{E^2(u)} - 1\right] \tag{2·4}$$

The best combination for (i,j) will be chosen according to the maximum value of the relative efficiency function defined by Murthy and Swartz [6] as

$$Eff(\hat{b}) = \frac{0.6079}{\phi(i, j, n)},$$

where

$$\phi(i, j, n) = n,\ var(\hat{b}) / b^2,$$

and 0.6079 be Cramer-Rao lower bound found by Menon [14]. Values of Eff($\hat{b}$) and best of (i,j) for different values of n are obtained by Murthty and Swartz [6] and these are shown in Table 1.

Best (i,j) are used to calculate the lower and upper bound for b, (Q_1, Q_2) and these are checked for n = 2 (1) 25 and α = 0.10, 0.05 and 0.01.

Using the fact that (2.1) is strictly decreasing function of u in the interval $(0, \infty)$, the shortest one sided confidence interval for b may be obtained by finding u* such that

$$p_r\{u < u^*\} = \int_0^{u^*} f(u)\ du = 1 - \alpha$$

After some simplification, u* is obtained as solution of

$$\{1 - \exp(-u^*)\} \sum_{r=0}^{i-1} \sum_{s=0}^{j-i-1} C_{rs} A^{-2} [(\rho + 1)\{\rho + \exp(-u^*)\}]^{-1} = \frac{1-\alpha}{k} \quad (2.5)$$

Since the LHS of (2.5) is an increasing function of u*, then it has only one real positive root, u*, in interval {0, ∞ }. This gives the one sided shortest confidence interval for b as {0, u*/-1 nz}.

Note that (2.5) is a special case from (2.2) when $u_0 = 0$ and $u_1 = u^*$

Best (i,j) are used also to calculate the upper bound of b, u* and these are checked for different value of n and α. Values of u_0, u_1 and u* are tabulated in Table 2 for the best choice of (i,j) from different values of n and α.

Point estimates of b according to (2.3) are calculated and compared with the estimates of b arising from (i) the maximum likelihood method and (ii) one and two sided shortest confidence intervals. These calculations are shown in Table 3 for some values of n.

3. Some Special Cases

(a) Setting i = 1 and j = 2, the following results are obtained
  (i) $f(u) = n(n-1) \exp(-u)/\{n-1+\exp(-u)\}^2$, u>0

which is the exact pdf. of $u = \hat{b}/b = \frac{-1}{b \ln z}$ obtained by Bain [4].

Note that Jaech [3] used the inverse of the ratio $t_{(2)}/t_{(1)}$ as a point estimator for b.
  (ii) u_0 and u_1 are given as solution of

$$u_0 = -\left[2 \ln (n-1) + u_1\right],$$

and

$$\frac{2(n-1)(1-\alpha)}{n} = (n-1)^2 \left(1 - \tfrac{1-\alpha}{n}\right) \exp(u_1) - \left(1 + \tfrac{1-\alpha}{n}\right) \exp(-u_1) \, .$$

$$(iii)\ u^* = \ln\left[\frac{(n-1)\alpha}{n-\alpha}\right] > 0, \textit{ for any } \alpha \textit{ and } n.$$

(b) Setting i = 1 and j = n, the following results are obtained

$$(i)\ f(u) = n(n-1)\sum_{s=0}^{n-2}\binom{n-2}{s}\frac{(-1)^{s}\exp(-u)}{(n-s-1)^{2}}\left[\frac{s+1}{n-s-1}+\exp(-u)\right]^{-2}, u > 0$$

(ii) u_0 and u_1 are given as solution of

$$\sum_{s=0}^{n-2}\binom{n-2}{s}\frac{(-1)^{s}}{(n-s-1)^{2}}\left[\exp(-u_0)\left\{\frac{s+1}{n-s-1}+\exp(-u_0)\right\}^{-2} - \exp(-u_1)\right.$$

$$\left.\left\{\frac{s+1}{n-s-1}+\exp)(-u_1)\right\}^{-2}\right] = 0$$

and

$$\left\{\exp(-u_0) - \exp(-u_1)\right\}\sum_{s=0}^{n-2}\binom{n-2}{s}\frac{(-1)^{s}}{(n-s-1)^{2}}$$

$$\left[\left\{\frac{s+1}{n-s-1}+\exp(-u_0)\right\}\left\{\frac{s+1}{n-s-1}+\exp)(-u_1)\right\}\right]^{-} = \frac{1-\alpha}{n(n-1)}.$$

(iii) u^* is given as solution of

$$\left\{1-\exp(-u^*)\right\}\sum_{s=0}^{n-2}\binom{n-2}{s}\frac{(-1)^{s}}{(n-s-1)}\left[\frac{s+1}{n-s-1}+\exp(-u^*)\right]^{-1} = \frac{1-\alpha}{n(n-1)}.$$

(c) Finally, setting i = 1 and j = i + 1, the following results are obtained

$$(i)\ f(u) = \binom{n}{i-1}\sum_{r=0}^{i-1}\binom{i-1}{r}\frac{(-1)^{r}}{(r+1)^{2}}\exp(-u)\left[\frac{n-i}{r+1}+\exp(-u)\right]^{-2}, u > 0$$

(ii) The values of u_0 and u_1 are given as solution of

$$\sum_{r=0}^{i-1}\binom{i-1}{r}\frac{(-1)^{r}}{(r+1)^{2}}\left[\exp(-u_0)\left(\frac{n-i}{r+1}+\exp(-u_0)\right)^{-2} - \exp(-u_1)\right.$$

$$\left.\left\{\frac{n-i}{r+1}+\exp(-u_1)\right\}^{-2}\right] = 0,$$

and

$$\{\exp(-u_0)-\exp(-u_1)\}\sum_{r=0}^{i-1}\binom{i-1}{r}\frac{(-1)^r}{(r+1)^2}\left[\left\{\frac{n-i}{r+1}+\exp(-u_0)\right\}\left\{\frac{n-i}{r+1}+\exp(-u_1)\right\}\right]^{-1}=\frac{1-\alpha}{\binom{n}{i-1}}.$$

(iii) u^* is obtained as solution of

$$\{1-\exp(-u^*)\}\sum_{r=0}^{i-1}\frac{(-1)^r\binom{i-1}{r}}{(r+1)(n-i+r+1)}\left[\frac{n-i}{r+1}+\exp(-u^*)\right]^{-1}=\frac{1-\alpha}{\binom{n}{i-1}}.$$

4. Numerical Example

As an illustrative of the use of the above results, consider a simulated life test on 25 components from a Weibull population with shape parameter b = 2.0. The observed failure times are as follows:

5.6787,	9.2687,	12.3363,	17.2963,	18.4013,	25.7988,
29.2903,	35.6573,	36.4914,	37.7244,	42.3065,	48.1906,
49.1258,	51.5104,	55.6767,	57.9729,	60.9938,	62.1711,
63.7520,	66.6626,	69.8409,	77.6757,	87.7077,	88.2826,
92.5132.					

These simulated failure times can be used to calculate the point estimates of b according to (2.3) for the best combination of (i,j) and different values of n. The optimum choices of (i,j) for n = 2 (1) 25 are shown in Table (1) and the results are listed in Tables 1 and 2.

For comparison, the MLE and the Murthy and Swartz [6] (2.3) formula for estimating b are listed in Table (3) with corresponding one and two sided shortest confidence intervals.

If (2.4) were applied to the simulated data, using the fifth and last observation, then the standard deviation of $\hat{b}$ is 0.373. Assuming normality of $\hat{b}$, the approximate 90%, 95% and 99% two sided confidence interval for b are respectively given as (0.538, 1.762), (0.419, 1.880) and (0.188, 2.112). Also the approximate upper 90%, 95% and 0.99% limit for b are respectively 1.665, 1.761 and 2.015.

Table 1. Optimum Choice of (i,j)

Sample size n	(i,j)	B(i,j,n) factor	Eff($\hat{b}$)	sample size n	(i,j)	B(i,j,n) factor	Eff($\hat{b}$)
2	1,2	1.442	0.43	14	3,14	0.728	0.69
3	1,3	0.962	0.55	15	3,15	0.702	0.69
4	1,4	0.788	0.59	16	3,16	0.682	0.69
5	1,5	0.696	0.59	17	4,17	0.754	0.69
6	2,6	0.956	0.60	18	4,18	0.732	0.70
7	2,7	0.874	0.62	19	4,19	0.712	0.70
8	2,8	0.804	0.63	20	4,20	0.694	0.70
9	2,9	0.750	0.65	21	4,21	0.680	0.70
10	2,10	0.710	0.65	22	4,22	0.664	0.70
11	3,11	0.834	0.66	23	5,23	0.714	0.70
12	3,12	0.792	0.67	24	5,24	0.700	0.70
13	3,13	0.756	0.68	25	5,25	0.686	0.70

Values of u_0, u_1 and u^* are calculated using IMSL subroutines for the best (i,j) and n=2 (1) 25. These values are shown in Table 2.

Table 2. Values of u_0, u_1, u^* which may be used to find a one and two sided shortest confidence interval for b with α=0.10, 0.5,0.01

n	α=0.10			α=0.05			α=0.01		
	u_0	u_1	u^*	u_0	u_1	u^*	u_0	u_1	u^*
2	-0.970	0.970	2.944	-1.033	1.033	3.664	-1.085	1.085	5.293
3	0.130	1.572	3.146	0.126	1.638	4.136	0.120	1.710	6.115
4	0.115	1.380	4.163	0.110	1.443	5.088	0.105	1.521	6.652
5	0.166	1.592	4.880	0.143	1.859	5.476	0.138	1.890	7.039
6	0.135	1.420	4.048	0.121	1.895	5.770	0.116	1.899	7.336
7	0.145	1.741	5.759	0.139	2.085	6.006	0.134	2.161	7.574
8	0.258	2.064	5.561	0.243	2.216	6.204	0.238	2.351	7.779
9	0.256	2.048	5.915	0.240	2.248	6.373	0.240	2.384	7.937
10	0.245	1.966	6.016	0.240	2.880	6.520	0.239	2.910	8.087
11	0.331	2.648	6.116	0.328	3.068	6.625	0.315	3.166	8.226
12	0.331	2.648	6.225	0.328	3.280	6.777	0.314	3.295	8.371
13	0.337	2.696	6.178	0.329	3.295	6.875	0.321	3.320	8.405
14	0.341	2.728	6.113	0.330	3.461	6.154	0.321	3.481	8.151
15	0.355	2.840	5.140	0.344	3.578	7.157	0.338	3.662	8.158
16	0.365	2.920	5.650	0.361	3.668	6.150	0.356	3.768	7.144
17	0.394	3.152	5.780	0.385	3.961	6.166	0.378	3.995	7.150
18	0.405	3.240	4.161	0.398	3.966	5.956	0.371	4.011	6.153
19	0.406	3.248	5.153	0.401	3.980	6.134	0.388	4.120	8.163
20	0.399	3.192	5.798	0.397	3.989	6.819	0.370	4.131	7.819
21	0.392	3.136	6.162	0.389	4.005	7.640	0.367	4.211	9.290
22	0.397	3.167	6.734	0.390	4.108	6.960	0.370	4.245	9.346
23	0.702	3.510	7.150	0.611	4.211	8.140	0.595	4.312	9.341
24	0.689	3.445	6.175	0.601	4.316	7.130	0.581	4.329	9.135
25	1.004	3.989	6.117	0.998	4.634	6.143	0.886	4.995	9.143

Table 3. Comparison between point and shortest confidence intervals for b

Type of Estimates / n	2	4	8	10	14	20	25
Murthy and Swartz formula	0.706	0.878	1.083	0.997	0.836	0.936	1.108
MLE	2.002	1.648	1.344	1.270	1.190	1.144	1.230
90% two sided SCI	(0,1.980)	(0.103,1.239)	(0.192,1.532)	(0.175,1.401)	(0.238,2.909)	(0.296,2.366)	(0.622,2.470)
95% two sided SCI	(0,2.108)	(0.099,1.296)	(0.180,1.645)	(0.171,2.052)	(0.231,2.422)	(0.294,2.967)	(0.618,2.869)
99% two sided SCI	(0,2.215)	(0.094,1.366)	(0.177,1.745)	(0.170,2.073)	(0.225,2.436)	(0.274,3.062)	(0.549,3.093)
90% one sided SCI	6.009	3.738	4.127	4.286	4.277	4.297	3.788
95% one sided SCI	7.479	4.568	4.605	4.645	4.306	5.054	3.804
99% one sided SCI	10.804	6.320	5.774	5.761	5.703	5.795	5.662

References

[1] Ashour, S.K. and Jones, P.W. "Shortest confidence intervals for the shape parameter of Weibull distribution based on two adjacent times". *Technometrics*, 19, (1977), 201-203.

[2] Jaech, J.L. "Estimation of Weibull distribution shape parameter when no more than two failures occur per lot". *Technometrics*, 6(4), (1964), 415-422.

[3] Bain, L.J. "Results for one or more independent censored samples from the Weibull or extreme-value distribution". *Technometrics*, 15, (1973), 279-288.

[4] Mann, N.R. "Estimators and exact confidence bounds for Weibull parameters based on a few ordered observations". *Technometrics*, 12(2), (1970), 345-361.

[5] Murthy, V.K. and Swartz, G.B. "Estimation of Weibull parameters from two-order statistics". *Journal of Royal Statistical Society, Series C*, 23(1), (1975), 96-102.

[6] Hassanein, K.M. Estimation of the parameters of the exterme value distribution by use of two or three order statistics. *Biometrika*, 56(2), (1969), 429-436.

[7] Bain, L.J. "Inferences based on censored samples from the Weibull or extreme-value distribution". *Technometrics*, 14, (1972), 693-702.

[8] Englehardt, M. and Bain, L.J. "Some complete and censored sampling results for the Weibull and extreme value distribution". *Technometrics*, 15, (1973), 541-549.

[9] Salem, S. "Estimation problems in Life Testing from grouped and non grouped data: Bayesian and non-Bayesian study". *Ph.D. Thesis*, Institute of Statistical Studies and Research, Cairo University. (1982). (Unpublished).

[10] Mood, A.M., Graybill, F.A. and Boes, D.C. *Introduction to the Theory of Statistics*. McGraw-Hill International Book Company, 3*rd* edition, 1974.

[11] Malik, H.J. and Trudel, R. "Probability density function of quatient of order statistics from the Pareto, power and Weibull distributions", *Commun. Statist. Theor. Meth.*, 11(7), (1982), 801-814.

[12] Guenther, W.C. "Shortest confidence intervals". *The American Statistician*, 23(1), (1969), 22-25.

[13] Akhlaghi, M.R. and Parsian, A. "A note on shortest confidence intervals". *Commun. Statist. Simula.*, 15(2), (1986), 425-433.

[14] Menon, M.V. "Estimation of the shape and scale parameters of the Weibull distribution". *Technometrics* 5, (1963), 175-182.

أقصر فترة ثقة من جانب واحد ومن جانبين لمعلمة الشكل لتوزيع ويبل بناء على أي احصائيتين لبيانات الفشل

عثمـان علـي شـلبي
أستاذ مشارك، قسم الأساليب الكمية ـ جامعة الملك سعود ـ الرياض ـ المملكة العربية السعودية
(قدم للنشر في ٢٠/٢/١٤١١هـ وقبل للنشر في ٢٨/١٢/١٤١١هـ)

ملخص البحث. البحث يتناول تقدير معلمة الشكل لتوزيع ويبل باستخدام أسلوب فترات الثقة القصيرة من جانب واحد ومن جانبين باستخدام احصائيتين مرتبتين لبيانات الفشل وذلك للعينات الصغيرة. وقد تمت تجربة النتائج التي حصلنا عليها على الحاسب الآلي باستخدام بيانات مولدة من توزيع ويبل.