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Estimation in Truncated Weibull Distribution

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Abstract. In this article we discuss the problem of estimating the parameters of the left and the right truncated Weibull distribution when truncation points are unknown, the estimation is based on type II censored data. Maximum and modified maximum likelihood estimation and variance-covariance matrix of estimators are derived.

Keywords: Truncated Weibull distribution; maximum likelihood; modified maximum likelihood; variance-covariance matrix.

1. Introduction

Truncated distributions arise when sample selection and/or observations is not available in some subregion of the sample space. This can occur as a consequence of actual elemination of part of the original data.

The present study is concerned with a truncated (left or right) Weibull distribution using data from censored type II sampling. The probability density function (p.d.f.) of the two parameter Weibull distribution with scale and shape parameters θ and b is given as

$$f(x;b,\theta) = (b/\theta)\, x^{b-1} \exp(-x^b/\theta) \qquad (1.1)$$

for x>0, b>0 and θ>0. The cumulative distribution function (c.d.f.) is given as

$$F(x;b,\theta) = 1 - \exp(-x^b/\theta) \qquad (1.2)$$

The mean (M'_x), variance (M_2) and the coefficient of skewness (α_3) of (1.1) are:

$$M'_x = \theta^{b'} \quad \Gamma_1 \quad ,$$

$$M_2 = \theta^{2b'} \quad (\Gamma_2 - \Gamma_1^2) \qquad (1.3)$$

and
$$\alpha_3 = \frac{\Gamma_3 - 3\Gamma_2\Gamma_1 + \Gamma_1^3}{(\Gamma_2 - \Gamma_1^2)^{3/2}}$$

Where

$\Gamma_k = \Gamma(1+kb')$, $b' = 1/b$ and $\Gamma(z)$ is gamma function defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$$

We assume that n units are put on test, and r is the number of failures observed in the duration of the experiment and $x_1 < x_2 < .. < x_r$ are the observed failure times. The joint distribution of $x_1, x_2 \ldots, x_r$ is defined as

$$L(\underline{X}, b, \theta) \simeq (b/\theta)^r \prod_{i=1}^{r} x_i^{b-1} \exp\{-\lambda(\underline{X},b)/\theta\} \tag{1.4}$$

Where

$$\lambda(\underline{X},b) = \sum_{i=1}^{r} x_i^b + (n-r)x_r^b$$

and (1.5)

$$\underline{X} = (x_i\,, i=1,2,..,r)$$

Now consider the following two cases: the first case when (1.1) is truncated on left at $x \in (T,\infty)$ and the second case when (1.1) is truncated on right at $x \in (O,T)$, the resulting truncated distributions has the following density functions:

$$f_l(\underline{X};T,b,\theta) = f(\underline{X};b,\theta)/\{1-F(T;b,\theta)\}, \quad x \geqslant T \tag{1.6}$$

for the left truncated

$$f_r(\underline{X};T,b,\theta) = f(\underline{X};b,\theta)/F(T;b,\theta), \quad 0 < x \leqslant T \tag{1.7}$$

for the right truncated
Where f(.) and F(.) are defined by (1.1) and (1.2) respectively.

Charernkavanich and Cohen [1] discussed this problem for complete samples with a variety of estimation problems involving truncated normal, gamma, Weibull, lognormal and various other truncated distributions. Bain and Weeks [2], Basu [3,4], Deemer and Votaw [5], Sathe and Varde [6], Yang and Sirvanci [7], and Suich and Rutemiller [8] give the main results in the case of truncated exponential distribution using censored data, and Ahmed [9] discussed the maximum likelihood method of estimation for the unknown parameters of truncated Weibull distribution with known truncation points. The present study is concerned with a Weibull distribution that is singly truncated at unknown truncation points with censored type II data. The truncation parameter plays a role similar to that played by the location paramter in the three parameter Weibull distribution.

2. Left Truncated Weibull Distribution

When the p.d.f.(1.1) is truncated on the left at x=T, the resulting p.d.f. (1.6) becomes

$$f_1(\underline{X};T,b,\theta) = (b/\theta)\, x^{b-1} \exp\{-(x^b - T^b)/\theta\},\ x \geq T, \tag{2.1}$$

The likelihood function of type II censored sample with p.d.f. (2.1) is given by

$$L_1(\underline{X};T,b,\theta) = \prod_{i=1}^{r} (n-i+1)\,(b/\theta)^r \prod_{i=1}^{r} x_{(i)}^{b-1} \exp\left[-\{\lambda(\underline{X};b) - nT^b\}/\theta\right] \tag{2.2}$$

Where $\underline{X}$ and $\lambda(\underline{X},b)$ are defined by (1.5). Now, to estimate the unknown parameters (T,b,θ) we will use the methods of maximum and modified maximum likelihood estimation.

2.1. The Method of Maximum Likelihood Estimation (MLE)

On taking logarithms of (2.2), differentiating and equating to zero, we have

$$\frac{\partial \ln L}{\partial \theta} = (-r/\hat{\theta}) + \{\sum_{i=1}^{r} x_{(i)}^{\hat{b}} + (n-r)x_{(r)}^{\hat{b}} - n\hat{T}^{\hat{b}}\}/\hat{\theta}^2 = 0,$$

$$\frac{\partial \ln L}{\partial b} = (r/\hat{b}) - \{\sum_{i=1}^{r} x_{(i)}^{\hat{b}} \ln x_i + (n-r)x_{(r)}^{\hat{b}} \ln x_{(r)} - n\hat{T}^{\hat{b}} \ln T\}/\hat{\theta}$$

$$+ \sum_{i=1}^{r} \ln x_{(i)} = 0$$

and

$$\frac{\partial \ln L}{\partial T} = n\,\hat{b}\hat{T}^{\hat{b}-1}/\hat{\theta} = 0 \tag{2.3}$$

The maximum likelihood estimators $\hat{\theta}$, $\hat{b}$ and $\hat{T}$ satisfy the following equations:

$$\hat{\theta} = (\sum_{i=1}^{r} x_{(i)}^{\hat{b}} + (n-r)\, x_{(r)}^{\hat{b}} - n\hat{T}^{\hat{b}})/r \quad ,\ r > 0,$$

$$\hat{b} = r\,(\sum_{i=1}^{r} x^{\hat{b}}_{(i)} + (n-r)\,x^{\hat{b}}_{(r)} - n\hat{T}^{\hat{b}})\,/\Big[r(\sum_{i=1}^{r} x^{\hat{b}}_{i}\ln x_i + (n-r)x^{\hat{b}}_{(r)}\ln x_{(r)} - n\hat{T}^{\hat{b}}\ln\hat{T}) - \sum_{i=1}^{r}\ln x_{(i)}\,(\sum_{i=1}^{r} x^{\hat{b}}_{(i)} + (n-r)\,x^{\hat{b}}_{(r)} - n\hat{T}^{\hat{b}})\Big] \quad (2.4)$$

where $\hat{T}$ is obtained such that $0 < T \leq x_1$.

Since the system of equations given in (2.3) does not yield explicit solutions for $\hat{\theta}$, $\hat{b}$ and $\hat{T}$, therefore putting $\hat{T} = x_1$ where x_1 is the first order statistics, then $\hat{\theta}$ and $\hat{b}$ may be obtained by solving the first two equations in (2.4)

The second equation of (2.4) can be solved numerically in $\hat{b}$ after replacing $\hat{T}$ with x_1. The estimate $\hat{\theta}$ then follows from the first equation of (2.4) after replacing estimates of $\hat{b}$ and $\hat{T}$.

2.2. The Method of Modified Maximum Likelihood Estimation (MMLE)

The modified estimators considered in this subsection were originally offered as alternates for maximum likelihood estimator of truncation point (T) because we can look to the role of truncation point in the two parameter Weibull distribution as that the role of location parameter in the three parameter Weibull and both distributions, complete three parameters Weibull distribution and the left truncated Weibull distribution (2.1) are identical except for the fact that the location parameter is replaced by T. For this case and since the maximum likelihood estimator of T were likely to be unsatisfactory; then several modified estimators to truncation point are considered by replacing the third equation of (2.3) with alternate functional relationships as was done by Cohen and whitten [10] in connection with parameter estimation in gamma and normal distribution.

2.2.1. The first modification (MMLE-1)

We use the first two equations of (2.3) and $\hat{T} = E(x_1)$, where E(.) is the usual expectation symbol with respect to the p.d.f. of x_1 from (2.1). Since we

$$E(x_1) = \theta'^{b'} \exp(T^b/\theta')\, I\,(T^b/\theta',\, 1+b') \quad (2.5)$$

Where $\theta' = \theta/n$, b' is defined in (1.3) and I (u,v) is defined by

$$I(u,v) = \int_{u}^{\infty} t^{v-1} e^{-t}\, dt$$

is the incomplete gamma function. Therefore, the first two equations of (2.4) and

$$\hat{T} = \hat{\theta}'^{\hat{b}'} \exp(\hat{T}^{\hat{b}}/\hat{\theta}')\, I\,(\hat{T}^{\hat{b}}/\hat{\theta}' , 1+\hat{b}')$$

are solved numerically to obtain $\hat{\theta}$, $\hat{b}$ and $\hat{T}$.

2.2.2. The Second modification (MMLE-2)

We use the first two equations of (2.3) and $E(x_1) = x_1$, therefore the first two equations of (2.4) and the function

$$\hat{T} = \hat{\theta}'^{\hat{b}'} \left[\ln x_{(1)} - \hat{b}' \ln \hat{\theta}' - \ln I(\hat{T}^{\hat{b}'} /\hat{\theta}' , 1+\hat{b}')\right]$$

are solved numerically to obtain $\hat{\theta}$, $\hat{b}$ and $\hat{T}$.

2.2.3. The third modification (MMLE-3)

We use the first two equations of (2.3) and $E(x_1) = \exp\{-(x_1^b - T^b)/\theta'\}$, therefore the first two equations of (2.4) and the function

$$1 = \hat{\theta}'^{\hat{b}'} \exp\,(x^{\hat{b}}_{(1)}/\hat{\theta}')\, I\,(\hat{T}^{\hat{b}} / \hat{\theta}' , 1+\hat{b}')$$

are solved numerically to obtain $\hat{\theta}$, $\hat{b}$ and $\hat{T}$.

2.2.4. The fourth modification (MMLE-4)

Newby [11] used the method of moments to estimate the location parameter of Weibull distribution from complete sample, so we modify this estimator of the location parameter to be equivalent to the case of left truncated Weibull distribution. Therefore T satisfy the following equation

$$\hat{T} = a_1 - \{ a_2/M_2(\hat{b})\}\, M_1'(\hat{b}) \tag{2.6}$$

Where

$$M_j'(\hat{b}) = \hat{\theta}^{j\hat{b}'} \exp\,(\hat{T}^{\hat{b}} / \hat{\theta}) I\,(\hat{T}^{\hat{b}} / \hat{\theta} \;,\; 1+j\hat{b}') \quad ,j\geq 1 \tag{2.7}$$

is the non-central moments of parent distribution,

$$M_j(\hat{b}) = \sum_{i=0}^{j} \binom{j}{i} (-M_1'(\hat{b}))^{j-i} M_j'(\hat{b}) \quad ,j\geq 2$$

is the central moments of parent distribution,

$$a_1 = \sum_{i=1}^{r} x_{(i)}/r, \qquad r > 0$$

is the sample mean of observed failures and

$$a_j = \sum_{i=1}^{r} (x_{(i)} - a_1)^j / r \quad , j \geqslant 2$$

is the sample central moment of observed failures.

Then the first two equations of (2.4) plus equation (2.6) are used to obtain a new estimate of θ, b and T using numerical method of iteration.

2.3. The Asymptotic Variance Covariance Matrix

The exact distribution of $\hat{\theta}$, $\hat{b}$ and $\hat{T}$ is not known explicity. However by using the general theory of MLE which shows that (θ,b,T) asymptotically has the multivariate normal distribution with mean vector μ′= (θ,b,T) and variance covariance matrix given by

$$\begin{bmatrix} E(-\dfrac{\partial^2 \ln L}{\partial \theta^2}) & E(-\dfrac{\partial^2 \ln L}{\partial \theta \partial b}) & E(-\dfrac{\partial^2 \ln L}{\partial \theta \partial T}) \\ & E(-\dfrac{\partial^2 \ln L}{\partial b^2}) & E(-\dfrac{\partial^2 \ln L}{\partial b \partial T}) \\ & & E(-\dfrac{\partial^2 \ln L}{\partial T^2}) \end{bmatrix}^{-1}$$

The exact expressions for the expectations in the above matrix are difficult to obtain. However, in practice we would need the estimate for variance covariance matrix which Cohen [12] recommended using the approximations

$$E(-\frac{\partial^2 \ln L}{\partial \theta \partial b}) = -\frac{\partial^2 \ln L}{\partial \theta \delta b}\Bigg|_{\theta=\hat{\theta},\, b=\hat{b}} , \text{ etc.}$$

These are given below

$$-\frac{\partial^2 \ln L}{\partial \theta^2} = \frac{2\sum x^{\hat{b}}_{(i)} + (n-r)\, x^{\hat{b}}_{(r)} - n\hat{T}^{\hat{b}}}{\hat{\theta}^3} - \frac{r}{\hat{\theta}^2} ,$$

$$-\frac{\partial^2 \ln L}{\partial\theta\partial b} = -\frac{\sum x^{\hat{b}}_{(i)} \ln x_{(i)} + (n-r)\, x^{\hat{b}}_{(r)} \ln x_{(r)} - n\hat{T}^{\hat{b}} \ln \hat{T}}{\hat{\theta}^2},$$

$$-\frac{\partial^2 \ln L}{\partial\theta\partial T} = n\, \hat{b}\, \hat{T}^{\hat{b}-1} / \hat{\theta}^2,$$

$$-\frac{\partial^2 \ln L}{\partial b^2} = \frac{r}{\hat{b}^2} + \frac{\sum x^{\hat{b}}_{(i)} \ln^2 x_{(i)} + (n-r)x^{\hat{b}}_{(r)} \ln^2 x_{(r)} - n\hat{T}^{\hat{b}} \ln^2 \hat{T}}{\hat{\theta}}$$

$$-\frac{\partial^2 \ln L}{\partial b\partial T} = -n\, \hat{b}\, \hat{T}^{\hat{b}-1} \ln \hat{T} / \hat{\theta} \qquad \text{and}$$

$$-\frac{\partial^2 \ln L}{\partial T^2} = n\, \hat{b}\, (\hat{b}-1)\, \hat{T}^{\hat{b}-2} / \hat{\theta}$$

Again under large sample theory of MLE , the approximat confidence intervals of θ , b and T are obtained using normal approximation to the MLE's.

If the modified maximum likelihood estimators given in section (2.2) are considered, then the approximate variance covariance matrix of $(\hat{\theta}, \hat{b})$ is given by

$$\begin{bmatrix} -\dfrac{\partial^2 \ln L}{\partial\theta^2} & -\dfrac{\partial^2 \ln L}{\partial\theta\partial b} \\ & -\dfrac{\partial^2 \ln L}{\partial b^2} \end{bmatrix}^{-1} \tag{2.7}$$

The variance of $\hat{T}$ for the first three modifications is

$$\mathrm{Var}(\hat{T}) = \hat{\theta}'^{2\hat{b}'} \exp(\hat{T}^{\hat{b}} / \hat{\theta}') \left[I(\hat{T}^{\hat{b}} / \hat{\theta}' , 1+2\hat{b}') - \exp(\hat{T}^{\hat{b}} / \hat{\theta}')\, I^2 (\hat{T}^{\hat{b}} / \hat{\theta}' , 1+\hat{b}') \right]$$

and for the fourth modification, the variance of $\hat{T}$ as indicated by Newby [11], is given by

$$\mathrm{Var}(\hat{T}) = \mathrm{Var}(a_1) + B^2\,\mathrm{Var}(a_2) + (AC)^2\,\mathrm{Var}(d_3) + 2\{B\,\mathrm{Cov}(a_1, a_2) + A\,B\,C\,\mathrm{Cov}(a_2, d_3) + A\,C\,\mathrm{Cov}(a_1, d_3)\} \qquad (2.8)$$

Where

$$A^{-1} \simeq R_3 \{ M_3^{-1}(\hat{b})\, dM_3(\hat{b})/d\hat{b} - \frac{1}{2} M_2^{-1}(\hat{b})\, dM_2(\hat{b})/d\hat{b} \},$$

$$B \simeq - M_1'(\hat{b}) / 2\,\hat{\theta}\, M_2(\hat{b}),$$

$$C \simeq \hat{\theta}\, \{ M_1'(\hat{b})\, M_2^{-1}(\hat{b})\, dM_2(\hat{b})/d\hat{b} - dM_1'(\hat{b})/d\hat{b} \},$$

$d_j = a_j / (a_2)^{\frac{1}{2}}$ is the standardized j^{th} sample central moment, $j \geqslant 3$,

$R_j = M_j(b) / \sqrt{M_2(b)}$ is the standardized j^{th} central moment of parent distribution, $j \geqslant 3$,

$$dM_j(\hat{b}) / d\hat{b} = -\, j\, M_j(\hat{b})\, \psi(1+j\hat{b}')\, \hat{b}^2 \qquad , j \geqslant 2,$$

$$dM_j'(\hat{b}) / d\hat{b} = -\, j\, M_j'(\hat{b})\, \psi(1+j\hat{b}')\, \hat{b}^2 \qquad , j \geqslant 1,$$

$\psi(.)$ is digamma function defined and tabulated by Abramowitz and Stegun [13],

$$\mathrm{Var}(a_1) = \hat{\theta}^2 M_2(\hat{b}) / r,$$

$$\mathrm{Var}(a_2) = \hat{\theta}^4 \{ M_4(\hat{b}) - M_2^2(\hat{b}) \} / r,$$

$$\mathrm{Var}(a_3) = \hat{\theta}^6 \{ M_6(\hat{b}) - 3M_2(\hat{b})\, M_4(\hat{b}) + 2M_2^3(\hat{b}) \} / r,$$

$$\mathrm{Var}(d_3) \simeq R_3^2 \{ [\mathrm{Var}(a_3) / z_3^2] + [9\mathrm{Var}(a_2) / 4z_2^2] - [3\mathrm{Cov}(a_2, a_3) / z_2 z_3] \},\quad z_j = \hat{\theta}^j M_j(\hat{b}),$$

$$\mathrm{Cov}(a_1, d_j) \simeq R_j \{ \mathrm{Cov}(a_1, a_j) / z_j - j\mathrm{Cov}(a_1, a_2) / 2z_2 \} / r,$$

$$\mathrm{Cov}(a_2, d_j) \simeq R_j \{ \mathrm{Cov}(a_2, a_j) / z_j - j\mathrm{Var}(a_2) / 2z_2 \} / r,$$

$$\mathrm{Cov}(a_2, a_j) \simeq (z_{j+2} - z_j z_2 - j z_{j-1} z_3) \qquad \text{and}$$

$$\mathrm{Cov}(a_1, a_j) \simeq (z_{j+1} - j z_{j-1} z_2) \qquad , j \geqslant 1 \text{ and } r > 0$$

3. Right Truncated Weibull Distribution

When the p.d.f. (1.1) is truncated on the right at x = T, the resulting truncated Weibull p.d.f. (1.7) becomes

$$f_r\,(\underline{X};\,T,b,\theta) = (b/\theta)\,x^{b-1}\exp(-x^b/\theta)\,/R(T) \quad , \quad 0 < x \leq T \tag{3.1}$$

and the likelihood function of type II censored sample with p.d.f. (3.1) is given by

$$L_r\,(\underline{X};\,T,b,\theta) = \prod_{i=1}^{r}(n-i-1)\,\{b/\theta R(T)\}^r \prod_{i=1}^{r} x_{(i)}^{b-1}\exp\{-\,\lambda(X;b)/\theta\} \tag{3.2}$$

Where $R(T) = 1-\exp(-T^b/\theta)$ and $\underline{X}$ and $\lambda(\underline{X};\,b)$ are defined in (1.5).

The unknown parameters (θ,b,T) will be estimated by using the methods of maximum and the modified maximum likelihood estimation.

3.1. The Method of Maximum Likelihood Estimation

The likelihood function (3.2) attains its maximum as $T \longrightarrow x_r$, where x_r is the last observation in a censored sample of size r out of n, so for the maximum likelihood estimators of (θ,b,T), we differentiate the natural logarithms of (3.2) to yield

$$\frac{\partial \ln L}{\partial \theta} = -\frac{r}{\theta} + \frac{r\,T^b\,e^{-T^b/\theta}}{\theta^2 R(T)} + \frac{\sum_{i=1}^{r} x_{(i)}^b + (n-r)\,x_{(r)}^b}{\theta^2}$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{r}{b} + \sum_{i=1}^{r}\ln x_{(i)} - \frac{r\,T^b \ln T\,e^{-T^b/\theta}}{\theta\,R(T)} - \quad \ldots.(3.3)$$

$$\{\sum_{i=1}^{r} x_{(i)}^b \ln x_{(i)} + (n-r)\,x_{(r)}^b \ln x_{(r)}\}\,/\theta$$

and

$$\frac{\partial \ln L}{\partial T} = -\,\frac{r\,b\,T^{b-1}\,e^{-T^b/\theta}}{\theta\,R(T)}$$

Equating the system of equations (3.3) with zero we can compute $\hat{\theta}$, $\hat{b}$ and $\hat{T}$ which are the solution of the set of equations

$$\hat{\theta} = \frac{r\hat{T}^{\hat{b}} + (e^{\hat{T}^{\hat{b}}/\hat{\theta}} - 1)\left(\sum_{i=1}^{r} x_{(i)}^{\hat{b}} + (n-r)x_{(r)}^{\hat{b}}\right)}{r(e^{\hat{T}^{\hat{b}}/\hat{\theta}} - 1)},$$

$$\hat{b} = \left[r\hat{T}^{\hat{b}} + r(e^{\hat{T}^{\hat{b}}/\hat{\theta}} - 1)\left(\sum_{i=1}^{r} x_{(i)}^{\hat{b}} + (n-r)x_{(r)}^{\hat{b}}\right)\right] /$$

$$\left[(e^{\hat{T}^{\hat{b}}/\hat{\theta}} - 1)\left(\sum_{i=1}^{r} x_{(i)}^{\hat{b}} \ln x_{(i)} + (n-r)x_{(r)}^{\hat{b}} \ln x_{(r)}\right) + \right. \quad \text{.....(3.4)}$$

$$\left. r\hat{T}^{\hat{b}} \ln \hat{T} - \sum_{i=1}^{r} \ln x_{(i)} \left\{ \hat{T}^{\hat{b}} + (e^{\hat{T}^{\hat{b}}/\hat{\theta}} - 1)\left(\sum_{i=1}^{r} x_{(i)}^{\hat{b}} + (n-r)x_{(r)}^{\hat{b}}\right)/r \right\} \right]$$

and $\hat{T}$ will be obtained such that $\hat{T} \geqslant x_{(r)}$.

Since the set of equations given in (3.4) do not have explicit solution for $\hat{\theta}$, $\hat{b}$ and $\hat{T}$, therefore if we substitute $\hat{T}$ with $x_{(r)}$ in the first two equations of (3.4), obtaining $\hat{\theta}$ and $\hat{b}$ is easier and their solution can be obtained using numerical techniques. Modified quasilinearization method for solving nonlinear equations works quite well, a brief description of this method and a summary of the algorithm is given by Wingo [14].

3.2. The Method of Modified Maximum Likelihood Estimation

In this subsection we will derive several modifications to the maximum likelihood estimators of T, replace the third equation in (3.3) with alternative functional relationships from θ, b and T.

3.2.1. The first modification

Replace $\partial \ln L/\partial T = 0$ with the relation $E(x_{(n)}) = \hat{T}$, where $E(x_{(n)})$ is the usual expectation symbol with respect to the p.d.f. of x_n from the right truncated Weibull distribution(3.1). However $\hat{\theta}$,$\hat{b}$ and $\hat{T}$ are obtained by solving the first two equations in (3.4) with $\hat{T}$ given by

$$\hat{T} = n(1-e^{-\hat{T}^{\hat{b}}/\hat{\theta}})^{-n}\sum_{i=0}^{n-1}\binom{n-1}{i}\frac{(-1)^i}{\hat{\theta}}\left(\frac{\hat{\theta}}{i+1}\right)^{\hat{b}'+1} G\left(\frac{\hat{T}^{\hat{b}}}{\hat{\theta}}(i+1),\, 1+\hat{b}'\right) \text{...(3.5)}$$

Where

$$G(u,v) = \int_0^u t^{v-1} e^{-t}\,dt$$

3.2.2. The second modification

Replace $\partial \ln L/\partial T = 0$ by the functional relationships $x_{(n)} = E(x_{(n)})$ where $E(x_{(n)})$ is defined by the right hand side of (3.5). However $\hat{\theta}$, $\hat{b}$ and $\hat{T}$ are obtained by solving the first two equations in (3.4) with $\hat{T}$ given by

$$\hat{T}^{\hat{b}} = -\hat{\theta} \ln \Big[1 - \Big\{ \frac{n}{x_{(n)}} \sum_{i=0}^{n-1} \binom{n-1}{i} \frac{(-1)^i}{\hat{\theta}} \Big(\frac{\hat{\theta}}{i+1}\Big)^{\hat{b}'+1} G\Big(\frac{\hat{T}^{\hat{b}}}{\hat{\theta}}(i+1),\, 1+\hat{b}'\Big) \Big\}^{1/n} \Big]$$

3.2.3. The third modification

Replace $\partial \ln L/\partial T = 0$ by the functional relationships $E(x_{(n)})$ equal to $F(x_{(n)})$ where $F(.)$ is the c.d.f. of n^{th} order statistic from (3.1), therefore $\hat{\theta}$, $\hat{b}$ and $\hat{T}$ are obtained by solving the first two equations in (3.4) with $\hat{T}$ given by

$$\hat{T}^{\hat{b}} = -\hat{\theta} \ln \Big[1 - \Big\{ n(1 - n(1-e^{-c})^{-1} \sum_{i=o}^{n-1} \binom{n-1}{i} \frac{(-1)^i}{\hat{\theta}} \Big(\frac{\hat{\theta}}{i+1}\Big)^{\hat{b}'+1} G\Big(\frac{\hat{T}^{\hat{b}}}{\hat{\theta}}(i+1),\, 1+\hat{b}'\Big) \Big\}^{1/(n-1)} \Big],\ c = x_{(n)}^{\hat{b}}/\hat{\theta}\,.$$

3.2.4. The fourth modification

Use the modified moment estimator defined in section (2.2.4) instead of $\partial \ln L/\partial T = 0$, therefore we have $\hat{T}$ satisfying (2.8) but $M_j'(\hat{b})$ is given by the following equation

$$M_j'(\hat{b}) = n(1-e^{-\hat{T}^{\hat{b}}/\hat{\theta}})^{-n} \sum_{i=0}^{n-1} \binom{n-1}{i} \frac{(-1)^i}{\hat{\theta}} \Big(\frac{\hat{\theta}}{i+1}\Big)^{j\hat{b}'+1} G\Big(\frac{\hat{T}^{\hat{b}}}{\hat{\theta}}(i+1),\, 1+j\hat{b}'\Big). \qquad (3.6)$$

3.3. The Asymptotic Variance Covariance Matrix

As discussed in section (2.3), the approximate variance covariance matrix is obtained by inverting Fisher's information matrix whose elements are given as:

$$-\frac{\partial^2 \ln L}{\partial \theta^2} = -(r/\hat{\theta}^2) + \left[\{2\sum_{i=1}^{r} x^{\hat{b}}_{(i)} + (n-r)\, x^{\hat{b}}_{(r)}\}/\hat{\theta}^3\right] + \{2r\hat{T}^{\hat{b}}/ \hat{\theta}^3 (e^{\hat{T}^{\hat{b}}/\hat{\theta}} - 1)\} - \{r\hat{T}^{2\hat{b}} e^{\hat{T}^{\hat{b}}/\hat{\theta}}/\hat{\theta}^4 (e^{\hat{T}^{\hat{b}}/\hat{\theta}} - 1)^2\},$$

$$-\frac{\partial^2 \ln L}{\partial \theta \partial b} = \{r\hat{T}^{\hat{b}} e^{\hat{T}^{\hat{b}}/\hat{\theta}} \hat{T}^{\hat{b}} \ln \hat{T} / \hat{\theta}^3 (e^{\hat{T}^{\hat{b}}/\hat{\theta}} - 1)^2\} - \{r\hat{T}^{\hat{b}} \ln \hat{T}/\hat{\theta}^2 (e^{\hat{T}^{\hat{b}}/\hat{\theta}} - 1)\} - \{\sum_{i=1}^{r} x^{\hat{b}}_{(i)} \ln x_{(i)} + (n-r)\, x^{\hat{b}}_{(r)} \ln x_{(r)}\}/\hat{\theta}^2 ,$$

$$-\frac{\partial^2 \ln L}{\partial \theta \partial T} = \{r\,\hat{b}\,\hat{T}^{2\hat{b}-1} e^{\hat{T}^{\hat{b}}/\hat{\theta}}/\hat{\theta}^3 (e^{\hat{T}^{\hat{b}}/\hat{\theta}} - 1)^2\} - \{r\,\hat{b}\,\hat{T}^{\hat{b}-1}/\hat{\theta}^2 (e^{\hat{T}^{\hat{b}}/\hat{\theta}} - 1)\},$$

$$-\frac{\partial^2 \ln L}{\partial b^2} = (r/\hat{b}^2) + \left[\{\sum_{i=1}^{r} x^{\hat{b}}_{(i)} \ln^2 x_{(i)} + (n-r)x^{\hat{b}}_{(r)} \ln^2 x_{(r)}\}/\hat{\theta}\right] - \{r\hat{T}^{2\hat{b}} \ln^2 \hat{T}\, e^{\hat{T}^{\hat{b}}/\hat{\theta}} / \hat{\theta}^2 (e^{\hat{T}^{\hat{b}}/\hat{\theta}} - 1)^2\} + \{r\,\hat{T}^{\hat{b}} \ln^2 \hat{T}/\hat{\theta} (e^{\hat{T}^{\hat{b}}/\hat{\theta}} - 1)\},$$

$$-\frac{\partial^2 \ln L}{\partial b \partial T} = -\{r\,\hat{b}\,\hat{T}^{\hat{b}-1} \ln \hat{T}\, e^{\hat{T}^{\hat{b}}/\hat{\theta}}/\hat{\theta}^2 (e^{\hat{T}^{\hat{b}}/\hat{\theta}} - 1)^2\} + \{r\,\hat{b}\,\hat{T}^{\hat{b}-1} \ln \hat{T}/\hat{\theta} (e^{\hat{T}^{\hat{b}}/\hat{\theta}} - 1)\}$$

and

$$-\frac{\partial^2 \ln L}{\partial T^2} = \{r\,\hat{b}^2\,\hat{T}^{\hat{b}-2}/(e^{\hat{T}^{\hat{b}}/\hat{\theta}} - 1)\} - \{r\,\hat{b}^2\,\hat{T}^{2\hat{b}-1)} e^{\hat{T}^{\hat{b}}/\hat{\theta}}/\hat{\theta} (e^{\hat{T}^{\hat{b}}/\hat{\theta}} - 1)^2\}.$$

If the modified maximum likelihood estimators given in section (3.2) are considered, the approximate variance covariance matrix of (θ, b) defined by (2.7) with elements given in the present section and variance of $\hat{T}$ for the first three modifications given by

$$\mathrm{Var}(\hat{T}) = E\,(x^2_{(n)}) - E^2(x_{(n)})$$

Where $E(x^j_{(n)})$ is defined by the right hand side of (3.6) and for the fourth modification, variance of $\hat{T}$ is defined by (2.8) but $M'_j\,(\hat{b})$ in (2.8) is replaced by (3.6).

As mentioned earlier, modified quasilinearization method for solving nonlinear equations works quite well.

4. Illustrative Examples

The practical application of estimators resulting in this work are illustrated with simulated data from the left and the right truncated Weibull distribution. So we will show the following two examples.

Example (1) Left Truncation

For this case, we have a random sample of size 50 out of 60 observations from a Weibull population in which $\theta = 1.5$, $b = 0.65$ and $T = 0.55$.

For this sample, MLE's and MMLE's calculations are summarized in Table 1. Approximate variances for the both MLE's and MMLE's are given also in Table 1. We note that from our data, the covariances between these estimators are less than 1.319×10^{-4}, which indicate that the resulting estimators in this work can be approximated by the normal distribution. Therefore, the asymptotic confidence intervals(c.i.) for the actual parameters are calculated and they are given in Table 1.

Table 1. Estimates from left truncated Weibull distribution

Methods of	θ		b		T	
estimation	Point	95% c.i.	Point	95% c.i.	Point	95% c.i.
	0.72266 (2.69×10^{-4})	(0.691,0.755)	2.17506 (1.44×10^{-5})	(2.168,2.183)	1.53830 (3.93×10^{3})	(1.415,1.661)
MLE					0.57190 (3.31×10^{-3})	(0.459,0.685)
	0.62399 (1.34×10^{-2})	(0.397,0.851)	1.614180 (1.07×10^{-3})	(1.550,1.678)	0.57381 (3.31×10^{-3})	(0.556,0.592)
MMLE. 1	0.62398 (1.37×10^{-2})	(0.395,0.853)	0.61418 (1.09×10^{-3})	(0.549,0.679)	0.57573 (3.31×10^{-3})	(0.558,0.594)
MMLE. 2	0.62398 (1.40×10^{-2})	(0.392,0.856)	0.61417 (1.10×10^{-2})	(0.409,0.820)	0.57120 (3.31×10^{-3})	(0.553,0.589)
MMLE. 3	0.62948 (1.34×10^{-2})	(0.396,0.852)	0.61717 (1.07×10^{-2})	(0.412,0.822)	0.67290 (1.86×10^{-4})	(0.646,0.701)
MMLE. 4	0.98738 (1.05×10^{-2})	(0.787,1.188)	0.89762 (1.83×10^{-3})	(0.812,0.982)		

Where numbers under the point estimates denotes to the variances of estimators.

Example (2) Right Truncation

In this case, we have a random sample of size 25 out of 40 observations from a Weibull population in which $\theta = 1.0$, $b = 1.6$ and $T = 1.7$. For this simulated data, MLE's and MMLE's are calculated and summarized in Table 2.

Asymptotic variances of the resulting estimators are given in Table 2 and the approximate confidence intervals of the parameters θ, b and T are calculated because we have the covariances between these estimators near to zero. Therefore and according to normal approximation of MLE's and MMLE's, calculation of c.i.'s are presented in Table 2.

Table 2. Estimates from right truncated Weibull distribution

Methods of estimation	θ		b		T	
	Point	95% c.i.	Point	95% c.i.	Point	95% c.i.
MLE	1.82276 (2.73×10^{-3})	(1.720,1.925)	3.97635 (3.57×10^{-5})	(3.965,3.990)	2.35870 (7.23×10^{-4})	(2.306,2.411)
	1.72639 (8.92×10^{-3})	(1.541,1.912)	2.68145 (1.84×10^{-4})	(2.655,2.708)	1.53861 (1.68×10^{-3})	(1.458,1.619)
MMLE. 1	1.97826 (2.69×10^{-2})	(1.657,2.300)	1.62543 (5.62×10^{-4})	(1.579,1.672)	1.75381 (1.83×10^{-3})	(1.700,1.838)
MMLE. 2	1.62987 (1.97×10^{-2})	(1.355,1.905)	1.99871 (9.72×10^{-2})	(1.388,2.610)	1.57901 (1.83×10^{-3})	(1.495,1.663)
MMLE. 3	1.62345 (6.24×10^{-2})	(1.134,2.113)	1.13045 (3.04×10^{-2})	(0.789,1.472)	1.67321 (1.83×10^{-3})	(1.589,1.757)
MMLE. 4	2.31452 (3.14×10^{-3})	(2.205,2.424)	1.51913 (1.19×10^{-3})	(1.452,1.587)	1.96320 (8.91×10^{-3})	(1.945,1.982)

Where numbers under the point estimates denotes to the variances of estimators.

It is of interest to note the following special cases:

(i) If $b = 1$, the left truncated Weibull distribution with p.d.f. (2.1) reduces to the well known two parameter exponential distribution in which T is now the location parameter.

(ii) The likelihood functions of censored type II when the parent distribution is the two parameter Weibull can be obtained as a special case from (2.2) if $T = 0$ and from (3.2) if $T = \infty$

(iii) Results of Charernkavanich and Cohen [1] can be obtained as a special case from the present work when $r = n$.

Estimation in the truncated Weibull distribution has proven to be somewhat more troublesome than estimation in the above special cases. However a FORTRAN program, with some modification, which is prepared by Wingo [14] handles these computations.

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تقدير معالم توزيع ويبل المبتور

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ملخص البحث . في هذا البحث ناقشنا مشكلة تقدير معالم توزيع ويبل المبتور من اليمين ومن اليسار وذلك عندما تكون نقطة البتر غير معلومة والتي سوف نعتبرها معلمة إضافية يجب تقديرها . وفي هذا البحث استخدمنا طريقة الإمكان الأعظم وطرقًا مختلفة معدلة للإمكان الأعظم . وتم اختبار النتائج على بيانات من عينة مراقبة مأخوذة من التوزيع المبتور على الحاسب الآلي وباستخدام لغة الفورتران .