J. *King Saud Univ.*, Vol. 12, *Admin. Sci.* (1), pp. 101-126, Riyadh (1420/2000)

A Dynamic Simultaneous Transportation Equilibrium Model

Mohamad K. Hasan
Department of International Transport and Logistics
College of Maritime Transport and Technology
Arab Academy for Science and Technology and Maritime Transport
Alexandria, Egypt
e-mail: mkhasan@hotmail.com

(Received 16-5-1417H; accepted for publication 16-2-1419H)

Abstract. A Dynamic Simultaneous Transportation Equilibrium Model (DSTEM) that can predict trip generation, trip distribution, modal split, and trip assignment on a transportation network at any period of time is developed. The DSTEM model is an extension (time dependant) of the static Simultaneous Transportation Equilibrium Model (STEM) developed by Safwat and Magnanti. We first formulated the STEM as a fixed-demand traffic assignment model by using a modified network representation where the basic network is augmented with virtual (dummy) links to represent several choice dimensions. We then expanded the augmented network to represent different time periods following the procedures suggested by Drissi-Kaitouni and Hameda-Benchekroun. The resulting fixed-demand traffic assignment formulation for the "expanded dynamic" network was then shown to be equivalent to the proposed DSTEM. Hence, the DSTEM formulation can easily be solved by any of the available methods for solving static fixed-demand traffic assignment models.

Introduction

Interest in dynamic transport models to estimate and forecast urban daily traffic flow patterns during short periods of time (e.g., peak periods) has been progressively increasing over the last decade for several legitimate reasons. First, it is believed that dynamic models would generally produce more accurate estimates and predictions compared with static models [1]. Second, the genuine interest in Intelligent Vehicle-Highway Systems (IVHS) made it more compelling to obtain and analyze real-time information on traffic flow patterns in urban areas and their temporal variations during peak periods. The use of dynamic models would certainly allow better utilization of the available real-time data. Third, urban traffic congestion is by its own nature a short-term dynamic phenomenon. The rate of traffic flow into the network increases over time until it reaches a peak rate which may be maintained for a certain period of time (usually short duration), then begins to decrease and so on for different times during the day at different locations on the network. A good review of research progress on dynamic transport models may be found in DK-HB [2] who summarized their review as follows: "The seventies have been a transition period between heuristic models (where the demand is assigned to instantaneous minimum cost paths), and optimization models that take into account the demand over the whole study horizon of time, but all of them incorporate important limitations (only one destination; unrealistic conditions on the cost functions so that the flow "reaches" the destination; possible violation of the link capacities, etc.)." In their paper, DK-HB [2] proposed a Dynamic Traffic Assignment Model which is mainly based on the assumption that the time spent by a vehicle on a link may be decomposed into a fixed travel time plus a waiting time. The fixed travel time corresponds to the free or uncongested travel time over the link. Then the vehicle is put in an exit queue (which resides on the same link) until it becomes possible to enter a forward link; this decision is based on the links costs and their capacities. They showed that their model leads to a dynamic network structure (a temporal expansion of the original (base) network, including the queues). This dynamic network is the adaptation of those networks of Ford and Fulkerson [3], Fulkerson [4], and Zwack and Thomposon [5]. They incorporated explicit link queues (rather than node queues in the Zwack and Thomposon's model). Therefore, according to their formulation the Dynamic Traffic Assignment Problem (DTAP) with fixed demand may be viewed as a "simple" Static Traffic Assignment Problem (STAP) over the expanded network. Hence all the methods developed for the STAP over the past 40 years may be used to solve the DTAP.

In this paper, we apply the same approach suggested by DK-HB [2] to a model which combines additional three dimensions with traffic assignment. The Dynamic Simultaneous Transportation Equilibrium Model (DSTEM) that we propose in this paper can predict trip generation, trip distribution, modal split, and traffic assignment simultaneously over a period of time (e.g., during peak periods). The DSTEM is a time-dependant extension of a static Simultaneous Transportation Equilibrium Model (STEM) which was recently developed by Safwat and Magnanti [1].

The static STEM model belongs to the class of transport equilibrium models which are cast as equivalent optimization problems. The first of such models is the elastic demand traffic assignment problem which appeared in the work of Beckmann et al [6]. In this problem, the number of trips between each origin-destination pair is a function of the travel time between that pair. Beckmann's model was cast as an equivalent optimization problem that when is being solved yields the desired transport equilibrium solution. This basic equivalent optimization formulation has several modeling enrichments. Evans [7] extended the formulation to include trip distribution, assuming fixed trip generation and an entropy model for trip distribution. Using the fact that an entropy distribution model implies a logit mode-split model, Florian and Nguyen [8] extended the formulation to include modal split. More recently, Safwat and Magnanti [1] further enriched the behavioral features of the equivalent optimization approach. In their model (i.e., the STEM) trip generation can depend upon the system's performance through an accessibility measure that is based on the random utility theory of users' behavior (instead of being fixed) and trip distribution is given by a more flexible logit model based on the random utility theory (instead of being given by a less flexible entropy model). In practice, the STEM model was applied to real-world transportation systems. These are the intercity passenger travel in Egypt (Safwat [9-11]) , the large-scale urban transportation network of Austin,Texas

(Safwat and Walton [12]), and the urban transportation network of Tyler,Texas (Hasan [13]). Moavenzadeh, Markow, Brademeyer and Safwat [14] included an extended version of the STEM model as a central component of a comprehensive methodology for intercity transportation planning in Egypt. This methodology has been used in several case studies involving multimodal transportation of passengers and freight in Egypt (Intercity Project [15]). The STEM model may be solved by globally convergent and efficient algorithms (Safwat and Brademeyer [16]).

In the next section of this paper, we describe the static STEM model and its formulation as a fixed-demand traffic assignment problem over a supernetwork. In section 3, we propose a Dynamic STEM (DSTEM) model. The DSTEM is formulated as a fixed-demand traffic assignment problem over an expanded supernetwork which represents the temporal variations of traffic flow patterns over time. Section 4 includes a summary and conclusions..

2.The Static Simultaneous Transportation Equilibrium Model (STEM)

In this section, we state the underlying assumptions of the STEM model and describe its formulation as a fixed demand equilibrium problem by using a modified network representation. In other words, the basic network is augmented with virtual (dummy) links to represent several choice dimensions. This augmented network is called a supernetwork. The supernetwork representation reduces the problem of solving trip generation, trip distribution, modal split, and trip assignment simultaneously (jointly) to that of finding the fixed demand user equilibrium flow pattern over a single supernetwork (see Sheffi [17; p. 231] for more details on the supernetwork concept).

We first introduce the following notations:

(N^b,A^b),	a basic network (i.e., any transportation network) consisting of a set N^b of nodes and a set A^b of links;
I ,	the set of origin nodes ($N^b \supseteq I$);
i ,	an origin node in the set I ;
D_i^b ,	the set of destinations that are accessible from a given origin i ($N^b \supseteq D_i^b$);
j ,	a destination in the set D_i^b ;
p ,	a simple (i.e., no node repeated) path in the network (N^b,A^b);

R^b, the set of origin-destination (O-D) pairs;

P^b_{ij}, the set of simple paths from origin i to destination j;

P^b, the set of simple paths in the network ($P^b = \cup\{ P^b_{ij} : I \in I, j \in D^b_i \}$)

a, a link in the set A^b;

f_a, flow on link a;

h_p, the flow on path p where $f_a = \sum_p \delta_{ap} h_p$

$$\delta_{ap} = \begin{cases} 1, & \text{if link a belongs to path p;} \\ 0, & \text{otherwise} \end{cases}$$

$C_a(f_a)$, the average cost (travel time)of transportation on link a;

C_p the average cost of transportation on path p from a given $i \in I$ to a given $j \in D^b_i$ where $C_p = \sum_{a \in A^b} \delta_{ap} C_a$;

u_{ij}, the average minimum "perceived" cost of transportation between origin i and destination j .

2.1 A STEM model

In this subsection, we give a brief description of a STEM model. For more details see Safwat and Magnanti [1].

$$G_i = \alpha S_i + E_i \qquad \forall i \in I$$

$$S_i = \max \{0, \mathrm{Ln} \sum_{k \in D^b_i} \exp(-\theta u_{ik} + A_k) \} \qquad \forall i \in I$$

$$T_{ij} = \frac{G_i \exp(-\theta u_{ij} + A_j)}{\sum_{k \in D_i^b} \exp(-\theta u_{ik} + A_k)} \qquad \forall\, ij \in R^b$$

$$C_p \quad \begin{cases} = u_{ij} & if \quad h_p > 0 \\ \geq u_{ij} & if \quad h_p = 0 \end{cases} \qquad \forall\, p \in P^b$$

$$C_p = \sum_{a \in A^b} \delta_{ap} C_a(f_a) \qquad \forall\, p \in P^b$$

where

G_i = the number of trips generated from origin i

T_{ij} = the number of trips distributed from origin i to destination j

S_i = an accessibility variable that measures the expected maximum utility of travel on the transport system as perceived from origin i

E_i = a composite measure of the effect that the socioeconomic variables, which are exogenous to the transport system, have on trip generation from origin i;

A_j = a composite measure of the effect that the socioeconomic variables, which are exogenous to the transport system, have on trip attraction at destination j ;

α = a parameter that measures the additional number of trips that would be generated from a given origin i if the expected maximum utility of travel, as perceived by travellers at i, increased by unity;

θ = a parameter that measures the sensitivity of the utility of travel between any given origin-destination pair ij due to changes in the system's performance between that given O-D pair.

The basic assumptions of this STEM model may be summarized as follows:

(1) Trip generation, G_i , is given by any general function as long as it is linearly dependent upon the system's performance through an accessibility measure, S_i, based on the random utility theory of travel behavior (i.e., the expected maximum utility of travel).

(2) Trip distribution, T_{ij}, is given by a logit model where each measured utility function includes the average minimum perceived travel cost, u_{ij}, as a linear variable.

(3) Modal split and trip assignment are simultaneously user optimized. Notice that the STEM framework allows for the modal split to be given by a logit model or to be system optimized together with trip assignment.

2.2 An Equivalent Convex Program (ECP) for STEM

Consider the following optimization problem

$$\text{Minimize} \sum_{a \in A} \int_{0}^{f_a} C_a(w_a)dw_a + \frac{1}{\theta} \sum_{i \in I} \left[\frac{\alpha}{2} S_i^2 + \alpha S_i - (\alpha S_i + E_i) \ln (\alpha S_i + E_i)\right]$$
$$+ \frac{1}{\theta} \sum_{ij \in R^b} [T_{ij} \ln T_{ij} - A_j T_{ij} - T_{ij}]$$

subject to

$$\sum_{j \in D_i^b} T_{ij} = \alpha S_i + E_i \qquad \forall i \in I$$

$$\sum_{p \in P_{ij}^b} h_p = T_{ij} \qquad \forall ij \in R^b$$

$$S_i \geq 0 \qquad \forall i \in I$$

$$T_{ij} \geq 0 \qquad \forall ij \in R^b$$

$$h_p \geq 0 \qquad \forall p \in P^b$$

$$f_a = \sum_p \delta_{ap} h_p \qquad \forall a \in A^b.$$

Safwat and Magnanti [1] proved that under mild assumption on performance functions and non-negativity and inequality assumption on demand parameters (i.e., $\theta>0$, $E_i \geq \alpha > 0$), the ECP has a unique solution that is equivalent to equilibrium on STEM.

2.3 The formulation of STEM as a Fixed Demand User Equilibrium (FDUE) Problem

In this Subsection, we show that by a modification of the basic network we can formulate the STEM as a FDUE problem.

The required modification is shown in Figure 1; the example network (consists of two origins and three destinations) is augmented by :

1) dummy links leading from each destination node jŒD_i^b to a dummy destination node, denoted by d_i^S Œ D^S (D^S the set of all destinations of the supernetwork, |I| =|D^S|).There is one dummy destination node associated with each origin node. The flow on each of the dummy links, from j to D^S , is T_{ij} and the equivalent travel time(cost) on each of these dummy links is $\frac{1}{\theta}$ [ln T_{ij} - Aj].

2) dummy links leading from each origin $i \in I$ to the dummy destination node $d_i^S \in D^S$. The flow on each of these dummy links, from i to d_i^S , is e_i=M_i-G_i where M_i is the maximum number of trips which may be generated from origin i. The travel time(cost) on each of these dummy links is $\frac{1}{\theta}$ [ln (M_i -e_i) - $\frac{1}{\alpha}$(M_i -E_i -e_i)].

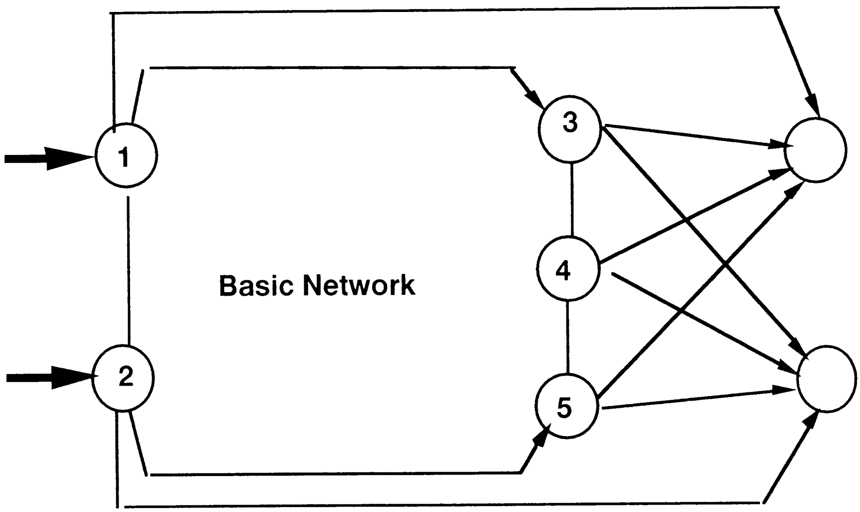

Fig. 1

Consider now a fixed demand user equilibrium problem defined over the modified supernetwork where the fixed demand is M_i which must be assigned between i and d_i^S for every i. This problem can be formulated as follows:

FDUE:

$$\text{Minimize} \quad Z(h,T,e)= \sum_{a\in A^b} \int_0^{f_a} C_a(w)\, dw + \frac{1}{\theta} \sum_{ij\in R^b} \int_0^{T_{ij}} (\ln(w) - A_j)\, dw$$

$$+\frac{1}{\theta} \sum_{i\in I} \int_0^{e_i} [\ln (M_i - w) - \frac{1}{\alpha}(M_i - E_i - w)]\, dw$$

subject to

$$\sum_{p\in P^b_{ij}} h_p = T_{ij} \qquad \forall\, ij \in R^b \quad (1) \quad (u_{ij})$$

$$\sum_{j\in D^b_i} T_{ij} + e_i = M_i \qquad \forall\, i \in I \quad (2) \quad (\gamma_i)$$

$$h_p \geq 0 \qquad \forall\, p\in P^b \quad (3) \quad (\omega_p)$$

$$T_{ij} \geq 0 \qquad \forall\, ij \in R^b \quad (4) \quad (\pi_{ij})$$

$$e_i \geq 0 \qquad \forall\, i \in I \quad (5) \quad (\lambda_i)$$

where

$$f_a = \sum_p \delta_{ap}\, h_p \qquad \forall\, a\in A^b.$$

It is easy to show that the FDUE problem is a convex program and has a unique solution under the same assumptions for ECP of STEM (i.e., $\theta>0$, $E_i \geq\alpha>0$). Next, we show that the Karush-Kuhn-Tucker optimality conditions for the FDUE problem are equivalent to STEM.

2.4 Equivalence between STEM and the optimality conditions of FDUE

Let u_{ij} for $ij \in R^b$, γ_i for $i \in I$, ω_p for $p\in P^b$, π_{ij} for $ij \in R^b$, and λ_i for $i \in I$ denote, respectively, the Karush-Kuhn-Tucker multipliers associated with the constraints (1) - (5) then the optimality conditions are given by the constraints (1) - (5) together with the following conditions:

$$C_p - u_{ij} - \omega_p = 0 \qquad \forall\, p \in P^b \quad (6)$$

$$\frac{1}{\theta}(\ln T_{ij} - A_j) + u_{ij} + \gamma_i - \pi_{ij} = 0 \qquad \forall\, ij \in R^b \quad (7)$$

$$\frac{1}{\theta}[\ln(M_i - e_i) - \frac{1}{\alpha}(M_i - E_i - e_i)] + \gamma_i - \lambda_i = 0 \qquad \forall\, i \in I \quad (8)$$

$$h_p\, \omega_p = 0 \text{ and } \omega_p \geq 0 \qquad \forall\, p \in P^b \quad (9)$$

$$T_{ij}\, \pi_{ij} = 0 \text{ and } \pi_{ij} \geq 0 \qquad \forall\, ij \in R^b \quad (10)$$

$$e_i\, \lambda_i = 0 \text{ and } \lambda_i \geq 0 \qquad \forall\, i \in I. \quad (11)$$

First, notice that the constraint (2) implies the trip generation conditions of STEM model. Since

$$\sum_{j \in D_i^b} T_{ij} + e_i = M_i$$

$$\sum_{j \in D_i^b} T_{ij} + (M_i - G_i) = M_i$$

$$\sum_{j \in D_i^b} T_{ij} = G_i = \alpha S_i + E_i\,. \quad (12)$$

Second, since $T_{ij} > 0 \ \forall\, ij \in R^b$ at the optimum solution, then $\pi_{ij} = 0 \ \forall\, ij \in R^b$.

Hence, (7) implies

$$T_{ij} = \mathrm{Exp}(-\theta\, \gamma_i)\, \mathrm{Exp}(-\theta\, u_{ij} + A_j). \quad (13)$$

From (8) we obtain

$$\frac{1}{\theta}\,[\,\ln(M_i - (M_i - G_i)) - \frac{1}{\alpha}(M_i - E_i - (M_i - G_i))\,] + \gamma_i - \lambda_i = 0$$

$$\frac{1}{\theta}\,[\,\ln(G_i) - \frac{1}{\alpha}(G_i - E_i)\,] + \gamma_i - \lambda_i = 0$$

$$\frac{1}{\theta}\,[\,\ln(\alpha S_i + E_i) - S_i\,] + \gamma_i - \lambda_i = 0$$

$$\mathrm{Exp}(-\theta\,\gamma_i) = (\alpha S_i + E_i)\,\mathrm{Exp}(-(S_i + \theta\,\lambda_i)). \tag{14}$$

Substituting the right hand side of (14) in (13) gives

$$T_{ij} = (\alpha S_i + E_i)\,\frac{\mathrm{Exp}(-\theta\,u_{ij} + A_j)}{\mathrm{Exp}(S_i + \theta\,\lambda_i)} \tag{15}$$

Summing (15) over all $j \in D_i^b$ and using (12) shows that

$$\mathrm{Exp}(S_i + \theta\,\lambda_i) = \sum_{j \in D_i^b} \mathrm{Exp}(-\theta\,u_{ij} + A_j)\,. \tag{16}$$

Therefore, at optimality (15) is given by

$$T_{ij} = (\alpha S_i + E_i)\,\frac{\mathrm{Exp}(-\theta\,u_{ij} + A_j)}{\sum_{j \in D_i^b} \mathrm{Exp}(-\theta\,u_{ij} + A_j)} \tag{17}$$

which is the logit trip distribution of STEM model.

Third, the optimality conditions (5), (11) imply that $\lambda_i = 0$ whenever $e_i > 0$ and therefore,(16) reduces to

$$S_i = \ln \sum_{j \in D_i^b} \mathrm{Exp}(-\theta\,u_{ij} + A_j)\,. \tag{18}$$

Fourth, we show that the optimum solution of FDUE implies a user optimized modal split and traffic assignment on STEM model. This results from optimality conditions (3), (6), (9) since $\omega_p = 0$ if $h_p > 0$ and, therefore,

$C_p = u_{ij}$. Moreover, if $h_p = 0$, then $\omega_p \geq 0$ and this implies $C_p = u_{ij} + \omega_p \geq u_{ij}$.

Thus, FDUE and STEM are equivalent.

3. A Dynamic Simultaneous Transportation Equilibrium Model (DSTEM)

In this section, we begin by stating the assumptions underlying the DSTEM model. We then describe the dynamic flow conservation constraints. Hence, we describe the corresponding temporal expanded network. Finally, we present the mathematical model. Our presentation closely follows that of Drissi-Kaitouni and Hameda-Benchekroun [2] and is based on the same assumptions mentioned in section 1, where we incorporated virtual (dummy) links to represent several choice dimensions for solving trip generation, trip distribution, modal split, and trip assignment simultaneously.

Before we describe our DSTEM model, we introduce the following notations for the supernetwork (N^S, A^S) shown in Figure 1, where N^S is the number of nodes and A^S is the number of links of this network.

We divide the set of nodes into three types:

(i) I, the set of all origin nodes.

(ii) D^b, the set of all destination nodes of the basic network $D^b = \cup \{ D_i^b, i \in I\}$.

(iii) D^S, the set of all dummy destination nodes d_i^S, $i \in I$. That is $N^S = N^b \cup D^S$.

Also, we divide the set of links A^S into three types :

(i) A^b, links of the basic network; these links carry the flow f_a $\forall a \in A^b$.

(ii) A^e, links that connect each origin $i \in I$ to the dummy destination $d_i^S \in D^S$; these links carry the flow e_i $\forall i \in I$.

(iii) A^{OD}, links that connect each destination $j \in D_i^b$ of the basic network to the dummy destination $d_i^S \in D^S$ $\forall i \in I$; these links carry the O-D flow T_{ij}, that is $A^S = A^b \cup A^e \cup A^{O\text{-}D}$.

3.1 Dynamic flow conservation constraints

Let A^S be the set of links of the supernetwork, and T be the time horizon (number of periods t =1,2,3,....,T). Thus

(i) for each $a \in A^b$ at each time period t , let

x_a^t : the total flow on link a at the beginning of period t,

u_a^t : the flow that enters link a during period t,

v_a^t : the flow that exits link a during period t,

q_a^t : part of the flow x_a^t ready to exit from link a at the beginning of period t,

τ_a : the fixed travel time on link a measured in number of periods of time.

Then, the constraint for link $a \in A^b$ at period t is

$$q_a^{(t+1)} = q_a^t + u_a^{(t-\tau_a)} - v_a^t \qquad \forall\, a \in A^b. \quad (19)$$

(ii) For each $a \in A^e$ at each time period t ,let

y_a^t : the total flow on a link a at the beginning of period t,

$\tilde{u}_a^t$: the flow that enters a link a during period t,

$\tilde{v}_a^t$: the flow that exits a link a during period t,

$\tilde{q}_a^t$: part of the flow y_a^t ready to exit from a link a at the beginning of period t,

λ_a : the fixed travel time on a link a measured in number of periods of time

Then, the constraint for a link $a \in A^e$ at period t is

$$\tilde{q}_a^{(t+1)} = \tilde{q}_a^t + \tilde{u}_a^{(t-\lambda_a)} - \tilde{v}_a^t \qquad \forall\, a \in A^e \quad (20)$$

(ii) For each $a \in A^{O\text{-}D}$ at each time period t ,let

z_a^t : the total flow on a link a at the beginning of period t,

$\hat{u}_a^t$: the flow that enters a link a during period t,

$\hat{v}_a^t$: the flow that exits a link a during period t,

$\hat{q}_a^t$: part of the flow z_a^t ready to exit from a link a at the beginning of period t,

μ_a : the fixed travel time on a link a measured in number of periods of time

Then, the constraint for link $a \in A^{O\text{-}D}$ at period t is

$$\hat{q}_a^{(t+1)} = \hat{q}_a^t + \hat{u}_a^{(t-\mu_a)} - \hat{v}_a^t \qquad \forall\, a \in A^{O\text{-}D}. \qquad (21)$$

(iv) Let M_i^t be the demand at origin node $i \in I$ during period t. M_i^t joins the queue Q_i^t at node i during period t, and let m_i^t be the part of that demand that enters the network during this period. Then, the constraint for the input flow is

$$Q_i^{(t+1)} = Q_i^t + M_i^t - m_i^t \qquad \forall\, i \in I. \qquad (22)$$

(v) The transfer of flow at nodes must satisfy the condition that the arriving flow is equal to the existing flow at any period of time t. This can be expressed by:

$$\sum_{a \in B_k^b} v_a^t + m_k^t = \sum_{a \in A_k^b} u_a^t + \tilde{u}_{a_k}^t \qquad \forall\, k \in I,\ a_k \in A^e \qquad (23)$$

$$\sum_{a \in B_k^b} v_a^t = \sum_{a \in A_k^b} u_a^t + \sum_{a \in A_k^{O\text{-}D}} \hat{u}_a^t \qquad \forall\, k \in D^b \qquad (24)$$

$$\sum_{a \in B_k^{O\text{-}D}} \hat{v}_a^t + \tilde{v}_{a_i}^t = O_k^t \qquad \forall\, k \in D^s,\ a_i \in A^e,\ i \in I \qquad (25)$$

$$\sum_{a \in B_k^b} v_a^t = \sum_{a \in A_k^b} u_a^t \quad \forall\ k \notin I\ , \forall\ k \notin D^b\ ,\ \forall\ k \notin D^S \tag{26}$$

where

B_k^b is the set of links in the basic network ($A^b \supset B_k^b$) that arriving at node k

A_k^b is the set of links in the basic network ($A^b \supset A_k^b$) that exiting from node k

$B_k^{O\text{-}D}$ is the set of links in the supernetwork ($A^{O\text{-}D} \supset B_k^{O\text{-}D}$) that arriving at node k

$A_k^{O\text{-}D}$ is the set of links in the super net work ($A^{O\text{-}D} \supset A_k^{O\text{-}D}$) that exiting from node k

O_k^t is the flow that exist super destination k.

3.2 Computation of μ_a , λ_a :

Step 1: Given $t_a^0 = \tau_a\ \forall\ a \in A^b$, compute u_{ij}^0(theminimumtraveltimebetweenoriginianddestinationj)for the basic network.

Step 2: Based on u_{ij}^0 , compute T_{ij}^0 as follows:

$$S_i^0 = \max\ \{0, \text{Ln} \sum_{k \in D_i^b} \exp\ (-\ \theta\ u_{ij}^0 + A_k)\ \} \quad \forall\ i \in I$$

$$G_i^0 = \alpha\ S_i^0 + E_i \qquad \forall\ i \in I$$

$$T_{ij}^{0} = \frac{G_i^0 \exp(-\theta u_{ij}^0 + A_j)}{\sum_{k \in D_i^b} \exp(-\theta u_{ik}^0 + A_k)} \qquad \forall\, ij \in R^b.$$

Step 3: The fixed travel time on O-D links is

$$\mu_a = \frac{1}{\theta}(\ln T_{ij}^0 - A_j) \qquad \forall a \in A^{O\text{-}D}.$$

Step 4: Compute $e_i^0 = M_i - G_i^0 \quad \forall\, i \in I$, then the fixed travel time on a link $a \in A^e$ is

$$\lambda_a = \frac{1}{\theta}\left[\ln(M_i - e_i^0) - \frac{1}{\alpha}(M_i - E_i - e_i^0)\right] \qquad \forall a \in A^e.$$

3.3 The Static Temporal Expanded Network (STEN)

In this subsection, we construct the temporal expanded network G = (A, N) that corresponds to the flow conservation constraints (19) - (26) of section 3.1.

Let θ_k^i be the length of the shortest path (shortest according to the free travel time) from the origin node n_i, $i \in I$, to node n_k, $k \in N^S$, and let $\theta_k^* = \min_{i \in I} \theta_k^i$, that is θ_k^* is the minimum time required to reach node n_k from any origin n_i, $i \in I$.

Thus, for $t < \theta_k^*$, no flow will enter node n_k. Now, we construct the network G as follows:

(1) Nodes:

Each node n_k, $k \in N^S$, of the supernetwork is expanded to (T+1) nodes n_k^t, $t = 0,1,2,\ldots,T$, $k \in N^S$, of the expended network G .

(2) Links:

(i) For each link $a = (n_i, n_j) \in A^b$ we construct, for each $t - \tau_a \geq \theta_j^*$

- one node l_a^t ,
- one link ($n_i^{(t-\tau_a)}$, l_a^t),
- one link (l_a^t , n_j^t) and
- one link (l_a^t , $l_a^{(t+1)}$) if $t < T$.

These three links correspond to the variables $u_a^{(t-\tau_a)}$, v_a^t and $q_a^{(t+1)}$ respectively and constraints (19) correspond to the (static) flow conservation constraints at the nodes l_a^t , $a \in A^b$, $t=0,1,2,\ldots,T$, such that $t - \tau_a \geq \theta_j^*$

(ii) For each link $a = (n_i , n_j) \in A^e$, $i \in I$, $j \in D^s$, we construct, for each $t - \lambda_a \geq \theta_j^*$

- one node $\tilde{l}_a^t$,
- one link ($n_i^{(t-\lambda_a)}$, $\tilde{l}_a^t$),
- one link ($\tilde{l}_a^t$, n_j^t) and
- one link ($\tilde{l}_a^t$, $\tilde{l}_a^{(t+1)}$) if $t < T$.

These three links correspond to the variables $\tilde{u}_a^{(t-\lambda_a)}$, $\tilde{v}_a^t$ and $\tilde{q}_a^{(t+1)}$ respectively, and constraints (20) correspond to the (static) flow conservation constraints at the nodes $\tilde{l}_a^t$, $a \in A^e$, $t=0,1,2,\ldots,T$, such that $t - \lambda_a \geq \theta_j^*$.

(iii) For each link $a = (n_i , n_j) \in A^{O\text{-}D}$, $i \in D^b$, $j \in D^s$, we construct, for each $t - \mu_a \geq \theta_j^*$

- one node $\hat{l}_a^t$,
- one link ($n_i^{(t-\mu_a)}$, $\hat{l}_a^t$),
- one link ($\hat{l}_a^t$, n_j^t) and

- one link $(\hat{l}_a^t, \hat{l}_a^{(t+1)})$ if $t < T$.

These three links correspond to the variables $\hat{u}_a^{(t-\mu_a)}$, $\hat{v}_a^t$ and $\hat{q}_a^{(t+1)}$ respectively, and constraints (21) correspond to the (static) flow conservation constraints at the nodes
$\hat{l}_a^t$, $a \in A^{O\text{-}D}$, $t=0,1,2,\ldots,T$, such that $t - \mu_a \geq \theta_j^*$.

(3) Origin nodes:

For each origin node n_i, $i \in I$, we construct, for each $t = 1,2,\ldots,T$:

- one node $\overset{L}{n}{}_i^t$,

- one link $(\overset{L}{n}{}_i^t, n_i^t)$ and

- one link $(\overset{L}{n}{}_i^t, \overset{L}{n}{}_i^{(t+1)})$ if $t < T$.

These two links correspond to the variables m_i^t and $Q_i^{(t+1)}$ respectively, and constraints (22) correspond to the (static) flow conservation constraints at the nodes $\overset{L}{n}{}_i^t$, $i \in I$ $t=0,1,2,\ldots,T$.

(4) Destination nodes:

For each super destination n_k, $k = d_i^S \in D^S$ we construct:

- one node r_k and

- one link (n_k^t, r_k) for each $t=0,1,2,\ldots,T$, such that $t \geq \theta_k^*$.

These links correspond to the variables O_k^t, $t=0,1,2,\ldots,T$, where r_k is augmented destination for all periods for super destination node n_k.

3.4 An illustrative example

Consider the simple basic network depicted in Fig. 2. This network includes one O-D pair (from node 1 to node 3) and two links. The fixed travel time on link 1 is $\tau_1 = 2$ and the fixed travel time on link 2 is $\tau_2 = 1$. The supernetwork representation of this network is shown in Fig. 3 where we added two links, one O-D link ,(i.e., link 3), connected node 3 to the super destination node $d_1^S = 4$, and the other link, (i.e., link 4), connected origin node 1 to the super destination node 4.

Now, we compute the fixed travel time on link 3 , i.e., μ_3 , and the fixed travel time on link 4. i.e., λ_4 . Let $\theta = .5$, $\alpha = 2.6$, $A_3 = 2$, and $E_1 = 11$, then

$$u^0_{13} = \tau_1 + \tau_2 = 2 + 1 = 3$$

$$S^0_1 = \ln [\exp (- \theta u^0_{13} + A_3) = \ln [\exp (-.5 (3) + 2)] = .5$$

$$G^0_1 = \alpha S^0_1 + E_1 = 2.6 (.5) + 11 = 12.3$$

$$T^0_{13} = 12.3$$

$$\mu_3 = \frac{1}{\theta} (\ln T^0_{13} - A_3) = \frac{1}{.5} (\ln (12.3) - 2) = 1.019 \approx 1.0$$

$$\lambda_3 = \frac{1}{\theta} [\ln (M_1 - e^0_1) - \frac{1}{\alpha} (M_1 - E_1 - e^0_1)] = \frac{1}{\theta} [\ln (G^0_1) - \frac{1}{\alpha} (G^0_1 - E_1)]$$

$$\lambda_3 = \frac{1}{.5} [\ln (12.3) - \frac{1}{2.6} (1.3)] = 4.0191 \approx 4.0$$

The links travel times are shown in Figure 3 on each link.

Figure 4 shows the corresponding STEN for the supernetwork in Fig. 3.

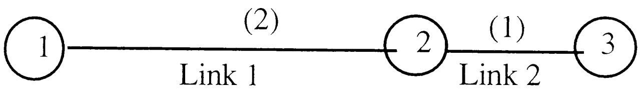

Fig. 2

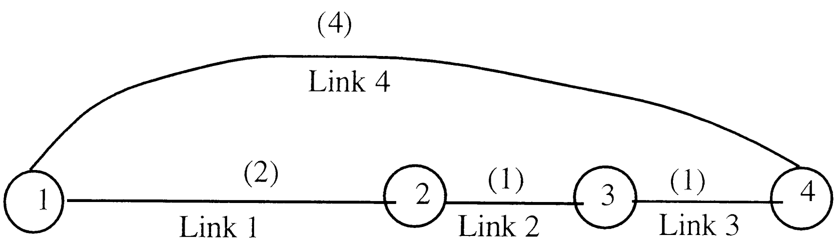

Fig. 3

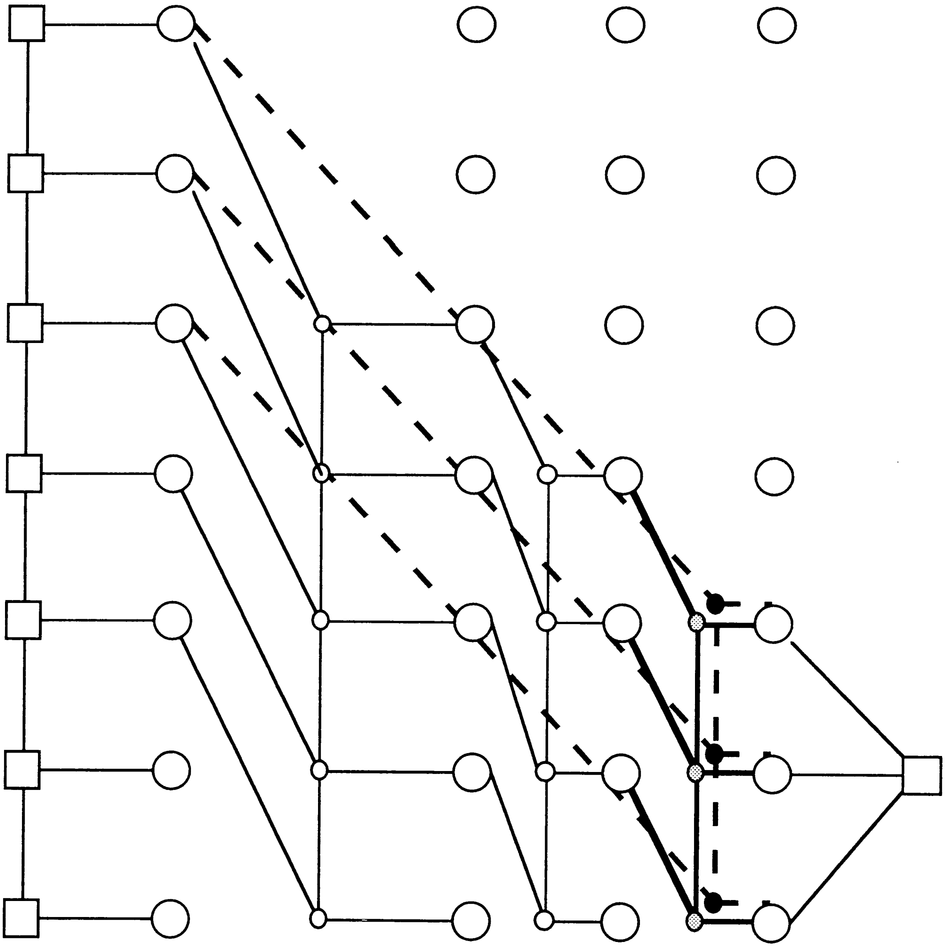

Fig. 4

3.5 Capacity constraints

1) For each link $a \in A^b$,

(i) The upper bound on the total link flow:

$$x_a^t \leq \tau_a C_a \qquad \forall a \in A^b, \ t = 0,1,2,\ldots,T$$

where C_a is the capacity of link a, expressed in vehicles/period. These constraints can be written as follows:

$$q_a^t + \sum_{j=t-\tau_a}^{t-1} u_a^j \le \tau_a \, C_a \qquad \forall\, a \in A^b \ , \ t = 0,1,2,\ldots,T. \qquad (27)$$

(ii) The upper bound on the entering flow:

$$u_a^t \le C_a \qquad \forall\, a \in A^b \ , \ t = 0,1,2,\ldots,T. \ (28)$$

(iii) The upper bound on the exiting flow:

$$v_a^t \le C_a \qquad \forall\, a \in A^b \ , \ t = 0,1,2,\ldots,T. \ (29)$$

2) For each link $a \in A^e$,

(i) The upper bound on the total link flow:

$$y_a^t \le \lambda_a \, \tilde{C}_a \qquad \forall\, a \in A^e \ , \ t = 0,1,2,\ldots,T$$

where $\tilde{C}_a = \tilde{C}_{a_i} = (M_i^* - E_i^*)$, $a_i \in A^e$, $i \in I$, is the maximum trip on link a_i, expressed in vehicles/period , M_i^* is the maximum number of trips which may be generated from node $i \in I$ expressed in vehicles/period, E_i^* is the minimum number of trips which can be generated from node $i \in I$ expressed in vehicles/period . These constraints can be written as follows:

$$\tilde{q}_a^t + \sum_{j=t-\lambda_a}^{t-1} \tilde{u}_a^j \le \lambda_a \, \tilde{C}_a \qquad \forall\, a \in A^e \ , \ t = 0,1,2,\ldots,T. \qquad (30)$$

(ii) The upper bound on the entering flow:

$$\tilde{u}_a^t \le \tilde{C}_a \qquad \forall\, a \in A^e \ , \ t = 0,1,2,\ldots,T. \qquad (31)$$

(iii) The upper bound on the exiting flow:

$$\tilde{v}_a^t \le \tilde{C}_a \qquad \forall\, a \in A^e \ , \ t = 0,1,2,\ldots,T. \qquad (32)$$

3) For each link $a \in A^{O\text{-}D}$,

(i) The upper bound on the total link flow:

$$y_a^t \leq \mu_a \hat{C}_a \qquad \forall\, a \in A^{O\text{-}D},\ t = 0,1,2,\ldots,T$$

where $\hat{C}_a = \hat{C}_{a_i} = M_i^*$, $a_i \in A^{O\text{-}D}$, $i \in I$, is the maximum number of trips which may be carried on link a_i, expressed in vehicles/period.These constraints can be written as follows:

$$\hat{q}_a^t + \sum_{j=t-\mu_a}^{t-1} \hat{u}_a^j \leq \mu_a \hat{C}_a \quad \forall\, a \in A^{O\text{-}D},\ t = 0,1,2,\ldots,T. \tag{33}$$

(ii) The upper bound on the entering flow:

$$\hat{u}_a^t \leq \hat{C}_a \qquad \forall\, a \in A^{O\text{-}D},\ t = 0,1,2,\ldots,T. \tag{34}$$

(iii) The upper bound on the exiting flow:

$$\hat{v}_a^t \leq \hat{C}_a \qquad \forall\, a \in A^{O\text{-}D},\ t = 0,1,2,\ldots,T. \tag{35}$$

3.6 The mathematical model

In this subsection, we formulate the mathematical model of DSTEM. The model is essentially a fixed demand user equilibrium traffic assignment model over the STEN network.

Consider STEN $\boldsymbol{G} = (A, \boldsymbol{N})$ and let M_i^t be the demand at period t from origin node $i \in I$ to destination node $j = d_i^s$ (r_j in the expanded network) ; M_i^t must be assigned on the paths $p \in P_{ij}^t$ where P_{ij}^t is the set of paths between the nodes n_i^t and node r_j in the STEN G. Hence, the static traffic assignment problem over STEN is:

$$\text{Minimize} \qquad \sum_{a \in A} \int_0^{f_a} S_a(w)\, dw$$

subject to

$$\underset{p\in P_{ij}^{t}}{h_p} = M_i^t \qquad \forall\ i \in I\ ,\ t=0,1,2,\dots\dots\ T$$

$$h_p \geq 0 \qquad \forall\ i \in I\ , \qquad \forall\ p\in P_{ij}^{t}\ ,\ t=0,1,2,\dots\dots T$$

$$f_a = \underset{i\in I}{}\ \overset{T}{\underset{t=0}{}}\ \underset{p\in P_{ij}^{t}}{}\ \delta_{ap}\, h_p \qquad \forall\ a\in A$$

and the capacity constraints (27)-(35), where

h_p is the flow on path $p\in P$, and P the set of all paths in ***G*** (i.e., $P=\cup\{\ P_{ij}^{t}\ :\ i \in I\ ,\ j \in D_i^S\ ,\ t=0,1,2,\dots\dots\ T\}$),

f_a is the flow on link $a\in A$,

S_a is the travel cost (time) on link $a\in A$ as a function of the flow (f_a) on that link.

If we refer to the unit travel cost on link $a\in A$ by the unit travel time, thus, when there is no congestion on the link, in general, this unit travel time is constant related to the length and the physical characteristics of the link. But when congestion is present, the unit travel time includes also a nonlinear delay component that depends on the flows and capacities of the links of the network. Therefore, in our model, we refer to the constant part by the fixed travel time on the link (i.e., τ_a , λ_a , or μ_a periods), whereas we refer to the nonlinear part of the link unit travel time by the time spend in the link queues (q_a^t , $a\in A^b$; $\tilde{q}_a^t$, $a\in A^e$; $\hat{q}_a^t$, $a\in A^{O\text{-}D}$; or Q_i^t , $i\in I$, $t=0,1,2,\dots\dots\ T$). The time spend by a vehicle in a queue q_a^t , $\tilde{q}_a^t$, $\hat{q}_a^t$, or Q_i^t is exactly one period of time after which it either exits the link, or joins the next link queue $q_a^{(t+1)}$, $\tilde{q}_a^{(t+1)}$, $\hat{q}_a^{(t+1)}$, or $Q_i^{(t+1)}$ in which another period of time will be spent, and so on. These unit travel costs lead to the following objective function:

$$\text{Minimize}\ \sum_{a\in A^b}\sum_{t=0}^{T}\int_0^{u_a^t}\tau_a\,dw+\sum_{a\in A^e}\sum_{t=0}^{T}\int_0^{\tilde{u}_a^t}\lambda_a\,dw+\sum_{a\in A^{O-D}}\sum_{t=0}^{T}\int_0^{\hat{u}_a^t}\mu_a\,dw$$

$$+\sum_{a\in A^b}\sum_{t=0}^{T}\int_0^{q_a^t}dw+\sum_{a\in A^e}\sum_{t=0}^{T}\int_0^{\tilde{q}_a^t}dw+\sum_{a\in A^{O-D}}\sum_{t=0}^{T}\int_0^{\hat{q}_a^t}dw$$

$$+ \sum_{i \in I} \sum_{t=0}^{T} \int_{0}^{Q_i^t} dw \qquad (36)$$

The capacity constraints are not often used as explicit constraints in network equilibrium models; instead capacities are implicitly expressed by a nonlinear component in the cost functions. Therefore,we will replace constraints (27)-(35) by a penalty term that we add to the objective function (36). Hence our model takes the following form:

$$\text{Minimize} \sum_{a \in A^b} \sum_{t=0}^{T} \tau_a u_a^t + \sum_{a \in A^e} \sum_{t=0}^{T} \lambda_a \tilde{u}_a^t + \sum_{a \in A^{O-D}} \sum_{t=0}^{T} \mu_a \hat{u}_a^t$$

$$+ \sum_{a \in A^b} \sum_{t=0}^{T} q_a^t + \sum_{a \in A^e} \sum_{t=0}^{T} \tilde{q}_a^t + \sum_{a \in A^{O-D}} \sum_{t=0}^{T} \hat{q}_a^t + \sum_{i \in I} \sum_{t=0}^{T} Q_i^t$$

$$+ \sum_{a \in A^b} \sum_{t=0}^{T} \nu \left[\max(0, u_a^t - C_a)^2 + \max(0, v_a^t - C_a)^2 + \max(0, x_a^t - \tau_a C_a)^2 \right]$$

$$+ \sum_{a \in A^e} \sum_{t=0}^{T} \nu \left[\max(0, \tilde{u}_a^t - \tilde{C}_a)^2 + \max(0, \tilde{v}_a^t - \tilde{C}_a)^2 + \max(0, y_a^t - \lambda_a \tilde{C}_a)^2 \right]$$

$$+ \sum_{a \in A^{O-D}} \sum_{t=0}^{T} \nu \left[\max(0, \hat{u}_a^t - \hat{C}_a)^2 + \max(0, \hat{v}_a^t - \hat{C}_a)^2 + \max(0, z_a^t - \mu_a \hat{C}_a)^2 \right]$$

subject to

$$\sum_{p \in P_{ij}^t} h_p = M_i^t \qquad \forall\ i \in I,\ t=0,1,2,\ldots\ldots, T$$

$$h_p \geq 0 \qquad \forall\ i \in I,\ \forall\ p \in P_{ij}^t,\ t=0,1,2,\ldots\ldots, T$$

$$f_a = \sum_{i \in I} \sum_{t=0}^{T} \sum_{p \in P_{ij}^t} \delta_{ap} h_p \qquad \forall\ a \in A$$

where ν is a large constant.

This is clearly a fixed-demand user equilibrium traffic assignment model whose solution satisfy the assumptions of the DSTEM proposed in this paper.

4. Conclusion

Dynamic transportation models have drawn much attention in recent years because they more accurately represent urban traffic patterns during peak periods. This is particularly important in view of the growing focus on IVHS and the need for better utilization of real-time traffic flow information.

DK-HB [2] formulated a dynamic traffic assignment model (DTAM) as a static fixed-demand traffic assignment problem over a temporal expanded network which when is solved would yield the desired solution for the DTAM. On the other hand, Safwat and Magnanti [1] developed a static STEM model which can predict trip generation, trip distribution, modal split and traffic assignment simultaneously on transportation networks. The STEM was formulated as an equivalent optimization problem which when solved would yield the desired transport equilibrium simultaneously.

In this paper we have developed a Dynamic STEM (DSTEM) which is essentially an extension of the static STEM following the procedures proposed by DK-HB [2] to develop their DTAM. First, the STEM was formulated as a fixed-demand traffic assignment problem defined on a supernetwork, then the DSTEM was formulated as an equivalent static traffic assignment problem defined on an expanded temporal network which can be easily solved by any of several available algorithms.

This DSTEM model should represent a significant enhancement of the predictive power of several dynamic transportation models. It is highly recommended to apply the DSTEM to real-life urban traffic congestion problems to demonstrate the enhanced accuracy of the model and hence its usefulness to be incorporated into the recent efforts to develop and apply advanced IVHS systems world-wide.

We would expect the network size to increase considerably and the model to inherent many of the advantages as well as the disadvantages of the STEM, DTAM, and many other related models. However, in view of the simplicity of the solution algorithm, the continuing advances in computational speed, and the reasonableness of its underlying assumptions, it is expected to be practical and desirable to implement the DSTEM to many real-life situations.

References

[1] Safwat, K. N. A. and Magnanti, T. L. " A Combined Trip Generation, Trip Distribution, Modal Split and Traffic Assignment Model."*Transportation Science*, 18, No. 2 (February 1988), 14-30.

[2] Drissi-Kaitouni, O. and Hameda-Benchekroun, A." A Dynamic Traffic Assignment Model and a Solution Algorithm." *Transportation Science* , 26, No. 2 (May 1992), 119-128.

[3] Ford, L. R. and Fulkerson, D. R. *Flows in Networks*. Princeton, New Jersey: Princeton University Press, 1962.

[4] Fulkerson, D. R. "Studies in Graph Theory, Part I."*Studies Math.* , 11 (1975).

[5] Zawack, D. J. and Thompson, G. L. " A Dynamic Space-Time Network Flow Model for City Traffic Congestion."*Transportation Science*, 21, No. 21 (August 1987), 153-162.

[6] Beckman, M.; McGuire, C. B. and Winston, C. B. *Studies in the Economic of Transportation*. New Haven, CT. : Yale University Press, 1956.

[7] Evans, S. P. "Derivation and Analysis of Some Models for Combining TripDistribution and Assignment."*Transportation Research*,10, No.1(1976), 37-57.

[8] Florian, M. and Nguyen, S. "A Combined Trip Distribution, Mode Split and Trip Assignment Model."*Transportation Research*, 12, No. 4 (1978), 241-246.

[9] Safwat, K. N. A. "The Simultaneous Prediction of Equilibrium on Large-scale Networks: A Unified Consistent Methodology for Transportation Planning." *Ph.D. Thesis*, Department of Civil Engineering, Massachusetts Institute of Technology , Cambridge, Massachusetts, 1982.

[10] Safwat, K. N. A. "Application of a Simultaneous Transportation Equilibrium Model to Intercity Passenger Travel in Egypt."*Transportation Research Record*, 1120 (1987), 52-59.

[11] Safwat, K. N. A. "Computational Experience with an Application of a Simultaneous Transportation Equilibrium Model to Intercity Passenger Travel in Egypt." *Transportation Research Record*, 1120 (1987), 60-67.

[12] Safwat, K. N. A. and Walton, C. M. "Computational Experience with an Application of a Simultaneous Transportation Equilibrium Model to Urban Travel in Austin,Texas. " *Transportation Research*, 22B, No. 6 (1988), 457-467.

[13] Hasan, M. K. "Comparative Analysis of Alternative Simultaneous TransportationNetwork Equilibrium Models." *Ph.D. Thesis*, Department of Urban and Regional planning, Texas A&M University, College Station, Texas, 1991.

[14] Moavenzadeh, F.; Markow, M.; Brademeyer, B. and Safwat, K. N. A. "A Methodology for Intercity Transportation Planning in Egypt." *Transportation Research*, 17A, No. 6 (1983), 481-491.

[15] Intercity Project Team. " Updating and Application of the Intercity Transportation Model." *Final Report. CU/MIT Technology Adaptation Program*, Development Research and Technological Planning Center, Cairo University, Egypt, 1986.

[16] Safwat, K. N. A. and Brademeyer, B. "Proof of Global Convergence of an Efficient Algorithm for Predicting Trip Generation , Trip Distribution, Modal Split and Traffic Assignment Simultaneously on Large-scale Networks." *International Journal of Computer and Mathematics with Applications*, 16, No. 4 (1988), 269-277.

[17] Sheffi, Y. *Urban Transportation Networks: Equilibrium Analysis with Mathematical Programming Methods*. Englewoods Cliffs, New Jersey: Prentice-Hall Inc.,1985.

نموذج آني ديناميكي لتوازن النقل

محمد كمال الدين أحمد حسن

قسم النقل والإمدادات الدولي، كلية النقل البحري والتقنية، الأكاديمية العربية للعلوم والتقنية والنقل البحري، الاسكندرية، جمهورية مصر العربية

(قدم للنشر في ١٦/ ٥/ ١٤١٧هـ ؛ وقبل للنشر في ١٦/ ١٢/ ١٤١٩هـ)

ملخص البحث. في هذه المقالة تم تطوير نموذج آني ديناميكي لتوازن النقل الذي يمكنه أن يتنبأ بتوليد الرحلات، توزيع الرحلات، فصل الرحلات حسب نوع وسيلة النقل المستخدمة وتخصيص الرحلات خلال أي فترة من الزمن. هذا النموذج هو امتداد (معتمدا على الزمن) لنموذج توازن النقل الآني الساكن الذي تم تطويره من قبل صفوت وماجنانتي. ولقد قمنا أولا بصياغة نموذج صفوت وماجنانتي إلى نموذج تعيين مرور ثابت الطلب باستخدام تمثيل شبكي معدل إذ أن شبكة النقل الأساسية تم دمجها مع وصلات صورية لتمثيل أبعاد متعددة لاختيار المستخدم للشبكة. ثم قمنا بتمديد هذه الشبكة المنتجة لتمثل فترات مختلفة للزمن حسب الإجراءات المقترحة من قبل دريس كاتوني وحميد بن شاكرون.

بعد ذلك بينا أن صياغة تخصيص المرور الثابتة الطلب الناتجة لشبكة الممتدة ديناميكيا تكافىء النموذج لنقل المقترح. وبناء على ذلك، فإن صيغة النموذج الآني الديناميكي لتوازن النقل المقترحة يمكن حلها سهولة باستخدام طرق عديدة متاحة لحل النماذج الساكنة لتخصيص المرور الثابت الطلب.