Optimal Production Run Size for a Multistage Manufacturing System*

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Abstract. In this paper, a mathematical model is presented to determine the optimal production run size and the corresponding optimal ordering quantity of raw material for a multistage manufacturing system. The optimal production run size and the optimal order quantity of raw material are determined simultaneously by incorporating the cost of holding WIP inventory, production setup cost, and the classical EOQ model for ordering raw material. In determining the optimal production run size, we have analyzed the average WIP inventory for two scenarios. In the first scenario, it is assumed that the last work station begins production before the first work station completes the production of all batches. On the other hand, the second scenario states that the first work station completes the production of all batches before the last work station begins production. By incorporating the cost of holding WIP inventory, production setup cost, and the classical EOQ model for ordering raw material, a more realistic total production cost function is achieved which leads to a better batch size and raw material ordering quantity.

Keywords: Work-in-process, Buffer size, Production run size, EOQ.

1. Introduction

Manufacturing systems consist of a set of connected work stations that carry out certain operations such as machining, forming, assembling, and inspection on raw materials, parts, and/or sub-components to produce a predesigned product. There are several types of manufacturing systems such as job shops, flow lines, continuous production systems, etc., that have been studied and analyzed from different points of view. The role of these production systems is to transform raw materials, parts and/or sub-components into a

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finished product which is delivered to customers. Therefore, the production system consists of several stages such as raw material inventory, processing, buffer (WIP) inventory, and final product inventory. Overall, the fundamental objective of manufacturing systems is the minimization of the total production costs.

Inventory is an essential component of a production system environment and plays an important role in manufacturing systems because it has a major impact on the total production costs and it accounts for a substantial portion of the production cost. As a result, manufacturing firms attempt to reduce the size and the cost of inventory. Inventory exists in these systems in three forms such as raw material, work-in-process (WIP), and finished good inventories. Raw material inventory includes items that require some type of processing to manufacture a final product. The purpose of raw material inventory is to allow a steady production of products by insuring parts availability. Some final products of a production system are considered as raw materials or sub-components to other production systems. On the other hand, WIP inventory exists in production systems where different types of raw materials are processed into finished products. Also, it exists as buffer between two work stations if the output of a work station is transferred in batches to the next work station. The objective of WIP inventory is to smooth and balance the work flow of operations in a production system. Finished product inventory represents items that are held at the manufacturing storage facility waiting to be shipped to the customers. Hence, inventory is used to balance variability and uncertainty in supply and demand. In general, the common objective of inventory is to decouple material flow between the production stages. However, as the size of inventory increases, the cost of managing and holding inventory increases as well. The costs that associate with holding inventory are as follow: storage space, handling, insurance, taxes, etc. Therefore, production systems aim at reducing the size of inventory in all the three forms of inventory to reduce the total production cost.

In this paper, a multistage production system is considered to determine the optimal production run size via minimizing the cost of holding raw material inventory, WIP inventory, raw material ordering cost, and production setup cost. Therefore, the overall production cost of a manufacturing system is minimized by incorporating the economic ordering quantity (EOQ) for raw material with holding cost of WIP inventory and production setup cost to determine the optimal production run size. Traditionally, EOQ and the cost of holding WIP inventory and production setup cost are determined separately. However, the optimum production run size here is determined by treating the EOQ, WIP inventory holding cost, and production setup cost as one entity of one system.

or the past few decades, many researchers have studied WIP inventory and its effects on the overall production costs. Ornek and Collier [1] develop a model to determine the size of the average WIP inventory and manufacturing lead time for multistage serial production systems. They have determined the average WIP inventory by dividing the time weighted WIP inventory on the time of one order cycle. They have considered the length of manufacturing cycle and the batch size as the main variables that are used to calculate the average WIP inventory and manufacturing lead time. Erel [2] has considered asynchronous serial lines in which WIP inventory is stored and transported in containers between work stations. He considers the number of containers, distances between work stations, and container size as the decision variables where the effect of these variables on the average throughput and WIP inventory is analyzed. He has developed an expression for the expected capacity of the buffer with non-zero transportation time and studied its relationship with the average throughput of the line. Sarker and Khan [3] have integrated the economic ordering quantity (EOQ) with the economic production quantity (EPQ) to minimize the total manufacturing cost for a two-stage manufacturing environment. They have analyzed the system for three cases. In the first case, they consider the ordering quantity of raw material to be equal to the requirement of the raw material for a lot of the production system. In the second case, they assume that the ordering quantity of the raw material is n times the quantity required to produce one lot of the product. In the last case, they assume that the ordering quantity of the raw material is n times the quantity required to produce one batch of the product. As a result, they have jointly determined the batch size for the product and order quantity of the required raw material of the product.

amasesh, Fu, Fong and Hayya [4] have presented an economic production lot size model that minimizes the total relevant cost when lot streaming is used. They have determined the optimal lot size for a single item in a multistage manufacturing system when a lot streaming is used and they have taken into account the cost of setup, transportation, holding WIP inventory, and holding finished product inventory. They have considered two cases. In the first case, they assume that the production rates are identical for all work stations. In the second case, they assume that the production rates are not identical for all work stations. However, they did not incorporate holding and ordering raw material costs in their model. Aldakhilallah [5] has analyzed WIP inventory level for a two-machine manufacturing system to determine the optimal batch size and the average WIP inventory. In determining the optimal batch size, the setup cost has been taken into account to evaluate the manufacturing system under two conditions. The first condition assumes that the processing time of the first machine is greater than the processing time of the second machine. The second condition states that the processing time of the first machine is less than the processing time of the second machine. In addition, each condition is analyzed assuming that setup procedure is performed once and more than once. Aldakhilallah [6] has studied the WIP inventory level for a two-stage manufacturing system to determine the maximum buffer size and the average WIP inventory assuming that setup time is negligible. The system is analyzed for three conditions and each condition is investigated for two cases which are based on the batch size. The first condition states that the processing time of the first work station is greater than that of the second work station. The second condition states that the processing time of the first work station is less than that of the second work station. The third condition states that the processing time of the first work station is equal to the processing time of the second work station. However, this paper differs from the previous research in that it analyzes the total production cost by incorporating the raw material ordering and holding cost with the cost of holding WIP inventory and production setup cost for a multistage production system.

This paper is organized as follows. Section 2 introduces definition and assumptions of the system under consideration. Section 3 determines the average WIP inventory for two scenarios and the optimal production run size for both scenarios. Section 4 provides a conclusion and direction for future research.

2. Definition and Assumption of the System

The production system considered in this paper is a multistage production system which consists of a set of connected work stations and a buffer space between every two work stations. Figure 1 depicts the production system that is being considered which consists of raw material inventory (RM), work stations (WS_j, j=1, 2, ..., m), WIP inventory (W_{jl} between work station j and work station l (l=j+1)), and finished goods inventory (FP). In this system, it is assumed that there is a single product to be produced. A work station processes a batch size of Q units, and then the batch is transferred to a buffer space between two work stations as WIP inventory. The WIP inventory size in the buffer space increases by one unit as soon as a work station processes a unit. The succeeding work station processes one unit of the batch per given time units. In other words, the number of units in each buffer space fluctuates up and down and the overall WIP inventory level begins to build up during the first operation and depletes during the last operation. Also, it is assumed that the total demand for the product is constant and the product is produced in equal batches. The processing time of operations is known and constant. Moreover, it is assumed that WS_{j+1} does not start production of a batch until the entire batch is processed by WS_j (where WS_j work station j, j=1, 2, . . . , m). In summary, the assumptions of the model are as follows:

1. A single product is considered.
2. The production system consists of m work stations.
3. Equal batches of Q units are produced at each work station.
4. The time it takes to transport one batch from one work station to the other is assumed to be negligible.
5. The production rate for each operation is known and constant.
6. No shortages are permitted.
7. All costs associated with the model are known.
8. Units transferred between work stations in batches.
9. The number of batches (n) and the batch size (Q) must be integers.

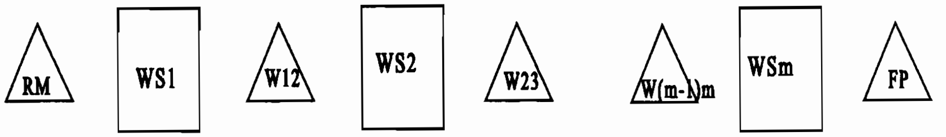

Fig. 1. A multistage production system.

The following notations are used throughout the paper:

- D: Demand of the product (units per time unit).
- δ_j: Amount of raw material j required to produce one unit of the product (j=1, 2, . . . , k).
- d_j: Demand of raw material j (j=1, 2, . . . , k) to produce the product ($d_j=\delta_j D$).
- Q: Production run size.
- h_w: Holding cost per unit per unit time of WIP inventory.
- λ_j: Holding cost per unit per unit time of raw material (j) inventory.
- q_j: Ordering quantity of raw material j ($q_j=\delta_j Q$).
- m: Number of work stations.
- S_j: Production setup cost for work station j (j=1, 2, . . . , m).
- α_j: Ordering cost of raw material j (j=1, 2, . . . , k).
- P_j: Processing time for work station j (j=1, 2, ... m).
- n: Number of batches.
- k: Number of type of raw materials required to produce the product.
- W_{jl}: WIP inventory level between work station j and work station l (l=j+1).

This production system is evaluated for the following condition: $P_1<P_2< \ldots <P_m$. This condition implies that the processing time of work station j is less than the processing time of work station j+1 (j=1, 2, . . . , m-1). In addition, this condition is considered and analyzed for two scenarios. The two scenarios are as follows:

Scenario I:

$$n P_1 Q > \sum_{j=1}^{m-1} P_j Q$$

Scenario II:

$$n P_1 Q < \sum_{j=1}^{m-1} P_j Q$$

In the following discussion, both scenarios will be analyzed and examples will be provided for illustration.

3. Mathematical Model

In this section, a total production cost function for the multistage production system described above is developed which takes into account the cost of holding WIP inventory, holding raw material inventory, production setup, and raw material ordering. The minimization of this total production cost function yields the optimal production run size and the corresponding optimal order quantities of raw materials. In addition, this function is developed and analyzed for two scenarios. The first scenario as given in Inequality (1) indicates that the last work station begins production before the first work station completes the production of all batches. On the other hand, the second scenario as given in Inequality (2) states that the first work station completes the production of all batches before the last work station begins production. Therefore, in the following discussion, the average WIP inventory, total production cost, and the optimal production run size will be determined for both scenarios.

3.1. Scenario I

In this scenario, the WIP inventory level behaves as shown in Fig. 2. It can be seen from the figure that WIP inventory level increases up to I_{max1} then increases again up to I_{max2} with a different slope. Furthermore, WIP inventory level will be depleted until it reaches zero level. For example, assume that we have a four-work station manufacturing system (m=4) and the WIP inventory level behaves as shown in Fig. 2. For the triangle A1 in Fig. 2, the slope of the line connecting O to I_{max1} is $1/P_1$ and the length of t_1 is $P_1Q+P_2Q+P_3Q$ which represents the production time of the first batch by the first three (in general, by m-1 work stations) work stations. In other words, it represents the time before the fourth (last) work station begins production. Therefore, the first maximum (I_{max1}) WIP inventory level will be determined as follows:

$$I_{max1}=(\text{slope})t_1=(1/P_1)(P_1Q+P_2Q+P_3Q). \quad (1)$$

Hence, the I_{max1} is as follows:

$$I_{max1}=Q+\frac{P_2Q}{P_1}+\frac{P_3Q}{P_1} \quad (2)$$

In addition, the second maximum (I_{max2}) WIP inventory level here is defined as follows:

$$I_{max2}=\text{Constant}+(\text{slope})(\text{the length of the line EF})$$

where the constant is I_{max1} and the slope is $(1/P_1-1/P_4)$. The length of line EF is defined as t_2-t_1 , where t_1 is $P_1Q+P_2Q+P_3Q$ and t_2 is nP_1Q which represents the production time needed by the first work station to complete the production of all batches of the product. As a result, I_{max2} is determined as follows:

$$I_{max2}=I_{max1}+(1/P_1-1/P_4)(nP_1Q-P_1Q-P_2Q-P_3Q) \tag{3}$$

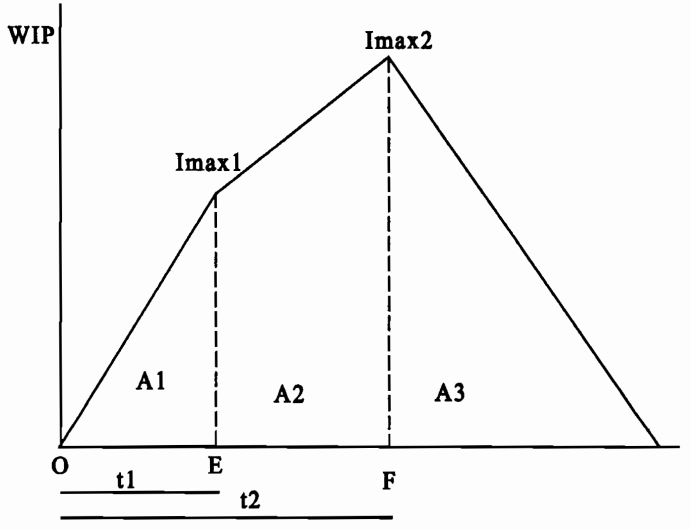

Fig. 2. The behavior of WIP inventory in Scenario I.

Therefore,

$$I_{max2}=Q+\frac{P_2Q}{P_1}+\frac{P_3Q}{P_1}+\left(\frac{1}{P_1}-\frac{1}{P_4}\right)(n\,P_1Q-P_1Q-P_2Q-P_3Q) \tag{4}$$

Simplifying Eq. (4) we have:

$$I_{max2}=n\,Q-\frac{(n-1)\,P_1Q}{P_4}+\frac{P_2Q}{P_4}+\frac{P_3Q}{P_4} \tag{5}$$

In addition, by dividing the area that represents the WIP inventory level in Fig. 2 to three separate areas as shown in the figure (A1, A2, and A3) and evaluating these areas we find that the average WIP inventory is as given in the following equation:

$$\overline{WIP} = \frac{1}{3}\left(\frac{I_{max1}}{2} + \frac{I_{max1} + I_{max2}}{2} + \frac{I_{max2}}{2}\right) \tag{6}$$

Substituting Eq. (3) and Eq. (5) into Eq. (6), we have:

$$\overline{WIP} = \frac{1}{3}\left(\frac{Q + \frac{P_2Q}{P_1} + \frac{P_3Q}{P_1}}{2} + \frac{Q + \frac{P_2Q}{P_1} + \frac{P_3Q}{P_1} + nQ - \frac{(n-1)P_1}{P_4} + \frac{P_2Q}{P_4} + \frac{P_3Q}{P_4}}{2} + \frac{nQ - \frac{(n-1)P_1}{P_4} + \frac{P_2Q}{P_4} + \frac{P_3Q}{P_4}}{2}\right) \tag{7}$$

Therefore, simplifying Eq. (7), we have:

$$\overline{WIP} = \frac{Q}{3}\left[(n+1) + \frac{P_2}{P_1} + \frac{P_3}{P_1} + \frac{P_2}{P_4} + \frac{P_3}{P_4} - \frac{(n-1)P_1}{P_4}\right] \tag{8}$$

The total production cost function (TC) of this production system which takes into account holding WIP inventory cost, production setup cost, raw material holding cost, and raw material ordering cost is as follows:

TC= (holding cost of WIP inventory)+(production setup cost)+(raw material ordering cost)+(holding cost of raw material).

The WIP inventory holding cost is represented by the average WIP inventory multiplied by the cost of holding one unit (h_w) as a WIP inventory. The production setup cost is represented by the number of batches need to be produced multiplied by the setup cost per batch per work station (S_j). In addition, the cost of holding and ordering raw material is represented by the classical EOQ model for each type of raw material, which is represented by the annual demand of raw material multiplied by ordering cost of raw material (α_j) and then divided by the order quantity of raw material plus the cost of holding raw material inventory. Therefore, the total production cost function is as given in the following equation:

$$TC=[\frac{Q}{3}((n+1)+\frac{P_2}{P_1}+\frac{P_3}{P_1}+\frac{P_2}{P_4}+\frac{P_3}{P_4}-\frac{(n-1)P_1}{P_4})]h_w+\sum_{j=1}^{m}\frac{DS_j}{Q}+\sum_{j=1}^{k}\frac{d_j\alpha_j}{q_j}+\sum_{j=1}^{k}\frac{q_j\lambda_j}{2} \quad (9)$$

Since we assume that $d_j=\delta_j D$ and $q_j=\delta_j Q$, then:

$$TC=(\frac{Q}{3}[(n+1)+\frac{P_2}{P_1}+\frac{P_3}{P_1}+\frac{P_2}{P_4}+\frac{P_3}{P_4}-\frac{(n-1)P_1}{P_4}])h_w+\frac{D}{Q}\sum_{j=1}^{m}S_j+\sum_{j=1}^{k}\frac{\delta_j D\alpha_j}{\delta_j Q}+\sum_{j=1}^{k}\frac{\delta_j Q\lambda_j}{2} \quad (10)$$

Also, since n=D/Q, then the total production cost as a function of Q can be written as follows:

$$TC=(\frac{Q}{3}[1+\frac{P_2}{P_1}+\frac{P_3}{P_1}+\frac{P_1}{P_4}+\frac{P_2}{P_4}+\frac{P_3}{P_4}]h_w+\frac{D}{3}(1-\frac{P_1}{P_4})h_w+\frac{D}{Q}\sum_{j=1}^{m}S_j+\frac{D}{Q}\sum_{j=1}^{k}\alpha_j+\frac{Q}{2}\sum_{j=1}^{k}\delta_j\lambda_j \quad (11)$$

Figure 3 shows the relationships between the elements of this cost function. As it can be seen from the figure, the total cost function is a convex function with a minimum value exists at the optimal production run size (Q*). Therefore, taking the first derivative of Eq. (11) with respect to Q and setting it to zero, we have:

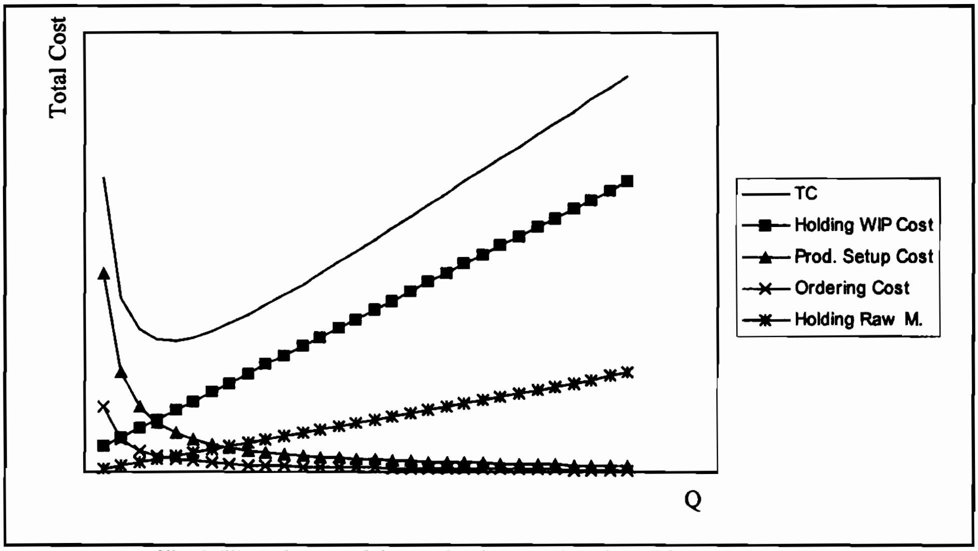

Fig. 3. The behavior of the production cost function (TC) for Scenario I.

$$\frac{\partial TC}{\partial Q}=\frac{1}{3}[1+\frac{P_2}{P_1}+\frac{P_3}{P_1}+\frac{P_1}{P_4}+\frac{P_2}{P_4}+P_3 overP_4]h_w$$

$$+0-\frac{D}{Q^2}\sum_{j=1}^{m}S_j-\frac{D}{Q^2}\sum_{j=1}^{k}\alpha_j+\frac{\sum_{j=1}^{k}\delta_j\lambda_j}{2}=0 \tag{12}$$

Solving this equation for Q^2, we have:

$$Q^2=\frac{6D(\sum_{j=1}^{m}S_j+\sum_{j=1}^{k}\alpha_j)}{2[1+\frac{P_2}{P_1}+\frac{P_3}{P_1}+\frac{P_1}{P_4}+\frac{P_2}{P_4}+\frac{P_3}{P_4}]h_w+3\sum_{j=1}^{k}\delta_j\lambda_j} \tag{13}$$

Hence, the optimal production run size (Q^*) is as follows:

$$Q^*=\sqrt{\frac{6D(\sum_{j=1}^{m}S_j+\sum_{j=1}^{k}\alpha_j)}{2[1+\frac{P_2}{P_1}+\frac{P_3}{P_1}+\frac{P_1}{P_4}+\frac{P_2}{P_4}+\frac{P_3}{P_4}]h_w+3\sum_{j=1}^{k}\delta_j\lambda_j}} \tag{14}$$

In addition, the corresponding optimal ordering quantities of raw materials are $q_j^*=\delta_j Q^*$ for j=1, 2, . . . , k. In general, the total production cost function and the corresponding optimal production run size for m work stations are given in Eq. (15) and Eq. (16), respectively, as follows:

$$TC=(\frac{Q}{3}[1+\frac{P_2}{P_1}+\frac{P_3}{P_1}+\ldots+\frac{P_{m-1}}{P_1}+\frac{P_1}{P_m}+\frac{P_2}{P_m}+\ldots+\frac{P_{m-1}}{P_m}]h_w$$

$$+\frac{D}{3}[1-\frac{P_1}{P_m}]h_w+\frac{D}{Q}\sum_{j=1}^{m}S_j+\frac{D}{Q}\sum_{j=1}^{k}\alpha_j+\frac{Q}{2}\sum_{j=1}^{k}\delta_j\lambda_j \tag{15}$$

$$Q^* = \sqrt{\frac{6D(\sum_{j=1}^{m} S_j + \sum_{j=1}^{k} \alpha_j)}{2[1+\frac{P_2}{P_1}+\frac{P_3}{P_1}+\ldots+\frac{P_{m-1}}{P_1}+\frac{P_1}{P_m}+\frac{P_2}{P_m}+\ldots+\frac{P_{m-1}}{P_m}]h_w+3\sum_{j=1}^{k}\delta_j\lambda_j}} \quad (16)$$

However, it has been assumed that both of the number of batches (n) and the batch size (Q) must be integers. We may obtain values of n and Q that are not integers which are considered inadmissible. Since TC is a convex function of Q, then we determine admissible values of n and Q using the following steps. Let Q represents the optimal batch size determined via Eq. (16) and n the corresponding number of batches. Step (1): select Q_j and Q_i which represent the nearest integer values that are higher and lower than Q, respectively. Step (2): determine n_j and n_i that correspond to Q_j and Q_i. Step (3): if Q_j and n_j and/or Q_i and n_i are integers, then go to Step (5). Step (4): set Q_j=Q_j+1 and Q_i=Q_i-1 and go to Step (2). Step (5): determine the total production cost and select the best integer values of both n and Q that give lowest cost. These steps will be illustrated in details in the following section.

3.1.1. An example

Assume that we have a manufacturing system which consists of 5 work stations. Table 1 gives the processing time and setup cost for each Ws_j, j=1, 2, .., 5, and data for the required raw materials assuming that there are three types of raw materials required to produce the product. Moreover, assume that the total demand (D) is 30 units, the cost of holding (h_w) one unit as a WIP inventory is $2. Figure 4 and Table 2 show the behavior of the WIP inventory level for a batch size (Q) of 5 units. Time in Table 2 refers to the completion time of a batch on work station 1 (WS_1) to work station 5 (WS_5). Also, WIP represents the total number of units per time of the WIP inventory between work stations (W_{ij}), which is the summation of WIP inventory in column 3, 4, 5 and 6. The last five columns (MC1-MC5) represent the number of units produced by each work station per time units.

Table 1. Data for the example

Work stations			Raw materials			
	Processing time	Setup cost		Type 1	Type 2	Type 3
WS_1	2.0	1.0	λ	1.5	0.8	1.0
WS_2	2.5	0.8	δ	2.0	1.0	3.0
WS_3	3.0	0.9	α	3.0	2.0	3.0
WS_4	3.5	1.5				
WS_5	4.0	1.2				

It can be seen from Fig. 4 and Table 2 that the level of WIP inventory reaches its first maximum level (I_{max1}) at time 55 ($P_1Q+P_2Q+P_3Q+P_4Q$). Also, using Eq. (3) we find that I_{max1} is 27.5 units. In addition, the level of WIP inventory reaches its maximum (I_{max2}) at time 60 (nP_1Q), I_{max2}=28.75 units. Hence, using Eqs. (15) and (16), we find that the optimal production run is 6.7207544 and the corresponding minimum production cost is 119.6294 and the number of batches (n) is 4.46378. However, Q and n are inadmissible and must to be integers. Therefore, we take the nearest integer values of Q until we obtain integer values of both Q and n. We start with Q=6 and Q=7 and check whether n is an integer or not. If n is still not integer, then we use Q=5 and Q=8, and so on until we obtain integer values for both Q and n that give the lowest TC. Hence, the best values of Q and n are 6 and 5, respectively, with a total production cost of 120.35. Figure 5 shows the total production cost curve for this example.

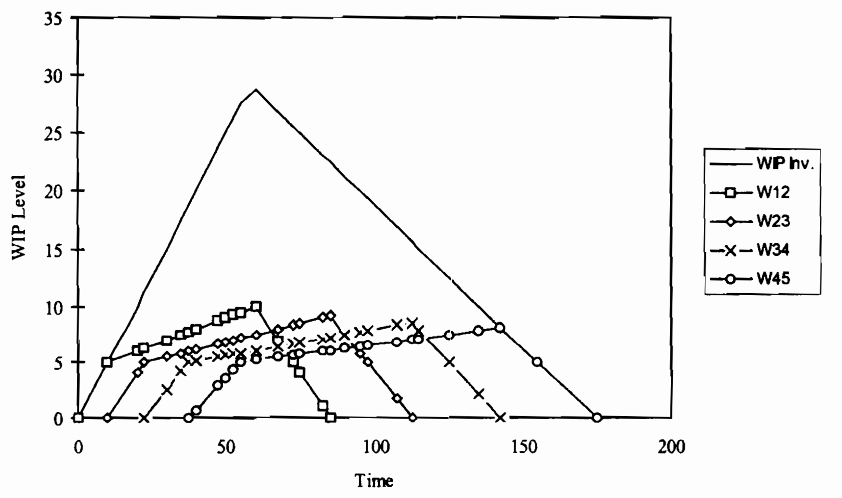

Fig. 4. The WIP inventory level for a 5-work station manufacturing system.

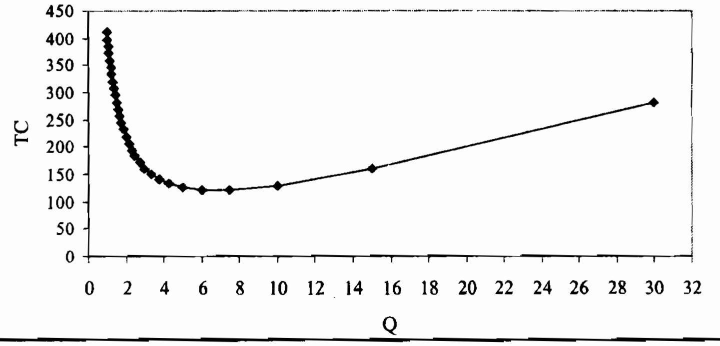

Fig. 5. The total production cost function of the example.

3.2. Scenario II

For this scenario, the variation of the WIP inventory level is described by Fig. 6. It can be seen from the figure that the WIP inventory level increases until it reaches its maximum at nQ units. The WIP inventory level increases at a rate of $1/P_1$ unit per time unit and it reaches its maximum at time t_1, where t_1 is represented by the time that the first work station completes the production of all batches (nQP_1). Thus, the slope of the line (ab) is $1/P_1$ and the length of the line (ac) is nQP_1. Then, the WIP inventory level stays at the maximum level until the last work station begins production in which the WIP inventory level decreases at a rate of $1/P_m$ unit per time unit. At this point, the time (t_2 to t_3) it takes for WIP inventory level to reach zero level is nQP_m which represents the production time of all batches by the last work station. Thus, the slope of the line (df) is $1/P_m$ and the length of the line (ef) is nQP_m. The WIP inventory at points (b) and (d) is given in the following:

at point (b) =(slope of the line ab)(the length of the line ac)=$(1/P_1)(nQP_1)$=nQ,

at point (d) =(-slope of the line df)(the length of the line ef)=$(1/P_m)(nQP_m)$=nQ.

Since the level of WIP inventory at both points is the same, we will use I_{max}=nQ to represent the maximum WIP inventory. Therefore, by dividing the area that represents the WIP inventory level in Fig. 6 to three areas as shown in the figure (B1, B2 and B3) and evaluating these areas we find that the average WIP inventory is as given in the following equation:

Table 2. An example of Scenario I for a five-work station manufacturing system

Time	WIP	W_{12}	W_{23}	W_{34}	W_{45}	MC1	MC2	MC3	MC4	MC5
0.00	0.00	0.00	0.00	0.00	0.00					
10.00	5.00	5.00	0.00	0.00	0.00	5.00				
20.00	10.00	6.00	4.00	0.00	0.00	5.00	4.00			
22.50	11.25	6.25	5.00	0.00	0.00	1.25	1.00			
30.00	15.00	7.00	5.50	2.50	0.00	3.75	3.00	2.50		
35.00	17.50	7.50	5.83	4.17	0.00	2.50	2.00	1.67		
37.50	18.75	7.75	6.00	5.00	0.00	1.25	1.00	0.83		
40.00	20.00	8.00	6.17	5.12	0.71	1.25	1.00	0.83	0.71	
47.50	23.75	8.75	6.67	5.48	2.86	3.75	3.00	2.50	2.14	
50.00	25.00	9.00	6.83	5.60	3.57	1.25	1.00	0.83	0.71	
52.50	26.25	9.25	7.00	5.71	4.29	1.25	1.00	0.83	0.71	
55.00	27.50	9.50	7.17	5.83	5.00	1.25	1.00	0.83	0.71	
60.00	28.75	10.00	7.50	6.07	5.18	2.50	2.00	1.67	1.43	1.25
67.50	26.88	7.00	8.00	6.43	5.45		3.00	2.50	2.14	1.88
72.50	25.63	5.00	8.33	6.67	5.63		2.00	1.67	1.43	1.25
75.00	25.00	4.00	8.50	6.79	5.71		1.00	0.83	0.71	0.63

Table 2. *(Contd.)*

Time	WIP	W_{12}	W_{23}	W_{34}	W_{45}	MC1	MC2	MC3	MC4	MC5
82.50	23.13	1.00	9.00	7.14	5.98		3.00	2.50	2.14	1.88
85.00	22.50	0.00	9.17	7.26	6.07		1.00	0.83	0.71	0.63
90.00	21.25	0.00	7.50	7.50	6.25			1.67	1.43	1.25
95.00	20.00	0.00	5.83	7.74	6.43			1.67	1.43	1.25
97.50	19.38	0.00	5.00	7.86	6.52			0.83	0.71	0.63
107.5	16.88	0.00	1.67	8.33	6.88			3.33	2.86	2.50
112.5	15.63	0.00	0.00	8.57	7.05			1.67	1.43	1.25
115.0	15.00	0.00	0.00	7.86	7.14				0.71	0.63
125.0	12.50	0.00	0.00	5.00	7.50				2.86	2.50
135.0	10.00	0.00	0.00	2.14	7.86				2.86	2.50
142.5	8.13	0.00	0.00	0.00	8.13				2.14	1.88
155.0	5.00	0.00	0.00	0.00	5.00					3.13
175.0	0.00	0.00	0.00	0.00	0.00					5.00

$$\overline{WIP} = \frac{1}{3}\left(\frac{I_{max}}{2} + \frac{I_{max} + I_{max}}{2} + \frac{I_{max}}{2}\right) \tag{17}$$

Substituting the value of I_{max} into Eq. (17) and simplifying it, we have:

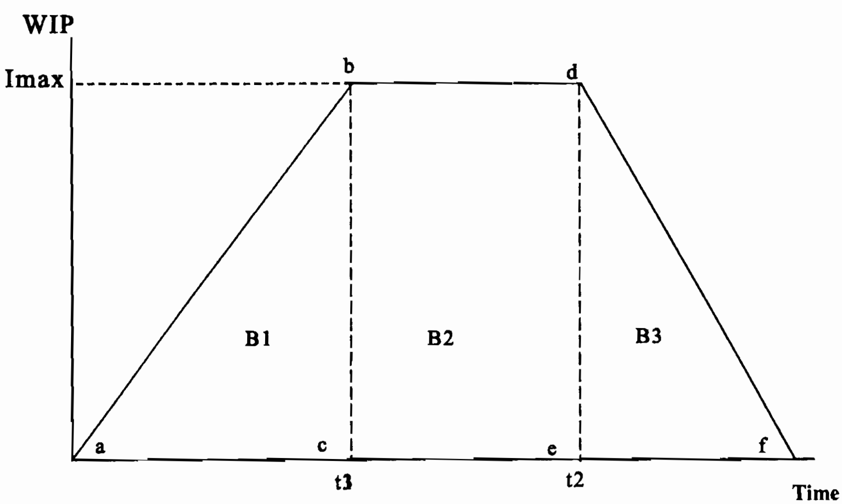

Fig. 6. The behavior of WIP inventory level in Senario II.

$$\overline{WIP} = \frac{2}{3} n Q \tag{18}$$

The total production cost function for this scenario is as follows:

TC= (holding cost of WIP inventory)+(production setup cost)+(raw material ordering cost)+(holding cost of raw material).

The WIP inventory holding cost is represented by the average WIP inventory multiplied by the cost of holding one unit (h_w) as a WIP inventory. The production setup cost is represented by the number of batches need to be produced multiplied by the setup cost per batch. In addition, the cost of holding and ordering raw material is represented by the classical EOQ model. Therefore, the total production cost function is as given in the following equation:

$$TC = \left(\frac{2nQ}{3}\right) h_w + \frac{D}{Q}\sum_{j=1}^{m} S_j + \sum_{j=1}^{k} \frac{d_j \alpha_j}{q_j} + \sum_{j=1}^{k} \frac{q_j \lambda_j}{2} \tag{19}$$

Since $d_j = \delta_j D$, $q_j = \delta_j Q$, and $n = D/Q$, then the total production cost as a function of Q can be written as follows:

$$TC = \frac{2Dh_w}{3} + \frac{D}{Q}\sum_{j=1}^{m} S_j + \frac{D}{Q}\sum_{j=1}^{k} \alpha_j + \frac{Q}{2}\sum_{j=1}^{k} \delta_j \lambda_j \tag{20}$$

Figure 7 describes the relationship between the elements of the total production cost function given in Eq. (20). As it can be seen from the figure, the cost of holding WIP inventory is constant over time. In addition, the total production cost function is clearly a convex function with a minimum value at the optimal production run size (Q^*) which can be determined by taking the first derivative of Eq. (20) with respect to Q and setting it to 0, we have:

$$\frac{\partial TC}{\partial Q} = -\frac{D}{Q^2}\sum_{j=1}^{m} S_j - \frac{D}{Q^2}\sum_{j=1}^{k} \alpha_j + \frac{\sum_{j=1}^{k} \delta_j \lambda_j}{2} = 0 \tag{21}$$

Solving Eq. (21) for Q^2, we have:

$$Q^2 = \frac{2D\left(\sum_{j=1}^{m} S_j + \sum_{j=1}^{k} \alpha_j\right)}{\sum_{j=1}^{k} \delta_j \lambda_j} \tag{22}$$

Hence, the optimal production run size (Q*) is as follows:

$$Q^* = \sqrt{\frac{2D\left(\sum_{j=1}^{m} S_j + \sum_{j=1}^{k} \alpha_j\right)}{\sum_{j=1}^{k} \delta_j \lambda_j}} \tag{23}$$

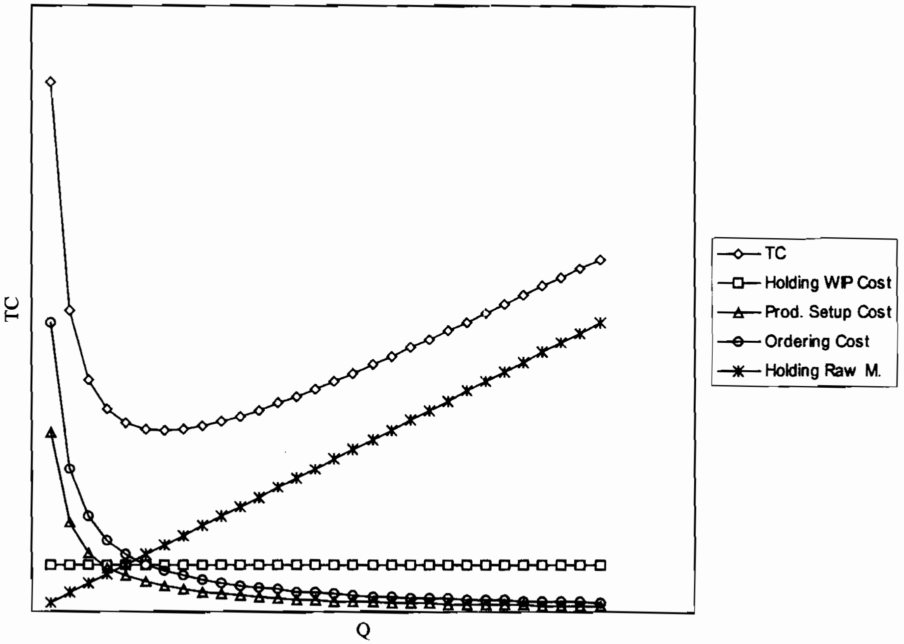

Fig. 7. The behavior of the production cost function (TC) for Scenario II.

In addition, the corresponding optimal ordering quantities of raw materials are $q_j^*=\delta_j Q^*$, ($j=1, 2, \ldots, k$). In general, since the total production cost function and the optimal production run size do not depend on the processing time of any work station and they depend only on the parameter values (D, S_f, α_j, δ_j, and λ_j, where $f=1, 2, \ldots, m$ and $j=1, 2, \ldots, k$), then Eqs. (20) and (23) are applicable to any manufacturing system environment as long as the assumptions of the model hold: 1) $P_1<P_2< \ldots <P_m$, and 2) Scenario II.

3.2.1. An example

Assume that we have a manufacturing system which consists of 5 work stations. Table 3 gives the processing time and setup cost for each work station, and data for the required raw materials. Also, assume that the total demand (D) is 30 units, the cost of holding (h_w) one unit as a WIP inventory is \$2.0. Figure 8 and Table 4 show the behavior of the WIP inventory level for a batch size (Q) of 5 units.

Table 3. Data for the example

Work stations				Raw material		
	Processing time	Setup cost		Type 1	Type 2	Type 3
WS_1	2.0	1.0	λ	1.5	1.4	1.2
WS_2	3.0	1.1	δ	3.0	2.0	3.0
WS_3	5.0	0.7	α	1.5	2.0	3.0
WS_4	6.0	1.2				
WS_5	8.0	1.3				

It can be seen from Fig. 8 and Table 4 that the level of WIP inventory reaches its maximum (I_{max}) at nQ (30) units at time nQP_1 at a rate of $1/P_1=1/2=0.5$. Then, it stays at this level until the last work station begins production in which WIP inventory level decreases at a rate of $1/P_5=1/8=0.12$. Hence, using Eqs. (20) and (23), we find that the optimal production run size and the corresponding minimum production cost and the number of batches are 7.193269, 128.41599, and 4.17057 respectively. However, Q and n are inadmissible and must be integers. We take the nearest integer values of Q until we obtain integer values of both Q and n. We begin with Q=6 and Q=10 and check whether n is an integer or not, and so on. Hence, the best values of Q and n are 10 and 3, respectively, with a total production cost of \$129.9. Figure 9 shows the total production cost function for this example.

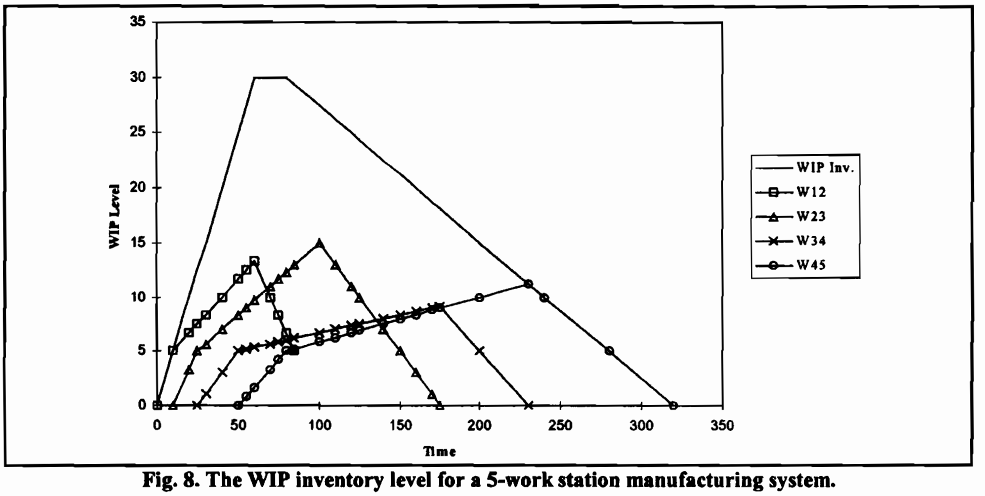

Fig. 8. The WIP inventory level for a 5-work station manufacturing system.

Table 4. An example of Scenario II for a five-work station manufacturing system

Time	WIP	W12	W23	W34	W45	MC1	MC2	MC3	MC4	MC5
0	0.00	0.00								
10	5.00	5.00	0.00			5.00				
20	10.00	6.67	3.33			5.00	3.33			
25	12.50	7.50	5.00	0.00		2.50	1.67			
30	15.00	8.33	5.67	1.00		2.50	1.67	1.00		
40	20.00	10.00	7.00	3.00		5.00	3.33	2.00		
50	25.00	11.67	8.33	5.00	0.00	5.00	3.33	2.00		
55	27.50	12.50	9.00	5.17	0.83	2.50	1.67	1.00	0.83	
60	30.00	13.33	9.67	5.33	1.67	2.50	1.67	1.00	0.83	
70	30.00	10.00	11.00	5.67	3.33		3.33	2.00	1.67	
75	30.00	8.33	11.67	5.83	4.17		1.67	1.00	0.83	
80	30.00	6.67	12.33	6.00	5.00		1.67	1.00	0.83	
85	29.38	5.00	13.00	6.17	5.21		1.67	1.00	0.83	0.63
100	27.50		15.00	6.67	5.83		5.00	3.00	2.50	1.88
110	26.25		13.00	7.00	6.25			2.00	1.67	1.25
120	25.00		11.00	7.33	6.67			2.00	1.67	1.25
125	24.38		10.00	7.50	6.88			1.00	0.83	0.63
140	22.50		7.00	8.00	7.50			3.00	2.50	1.88
150	21.25		5.00	8.33	7.92			2.00	1.67	1.25
160	20.00		3.00	8.67	8.33			2.00	1.67	1.25
170	18.75		1.00	9.00	8.75			2.00	1.67	1.25
175	18.13		0.00	9.17	8.96			1.00	0.83	0.63
200	15.00			5.00	10.00				4.17	3.13
230	11.25			0.00	11.25				5.00	3.75
240	10.00				10.00					1.25
280	5.00				5.00					5.00
320	0.00				0.00					5.00

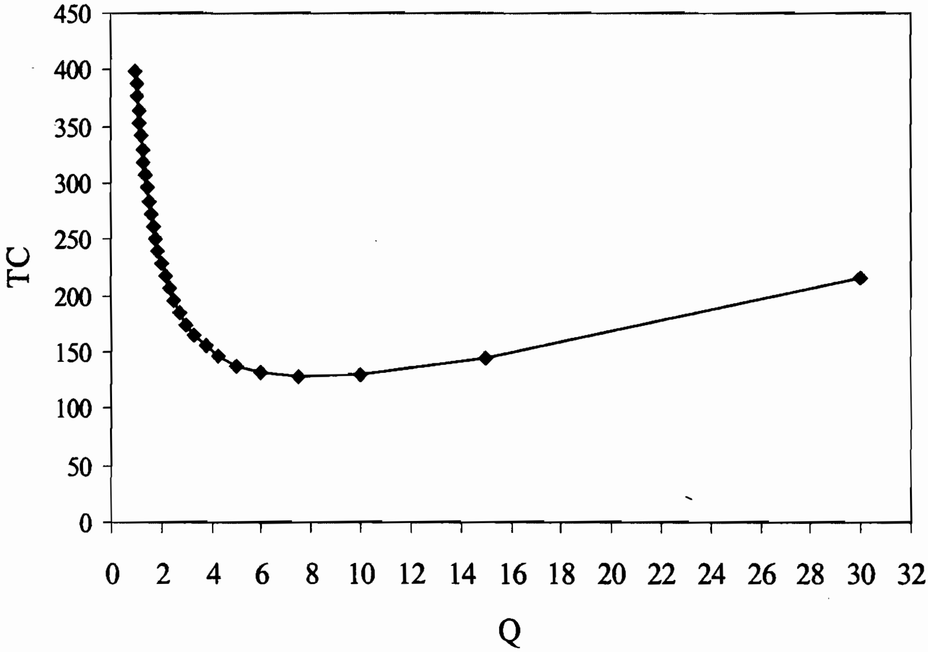

Fig. 9. The total production cost function of the example.

4. Conclusion

In this paper, we have developed a mathematical model to derive the optimal production run size and the complement optimal ordering quantities of raw materials for a multistage manufacturing system. In determining the optimal production run size, we have analyzed the average WIP inventory for two scenarios. In the first scenario, it is assumed that the last work station begins production before the first work station completes the production of all batches. On the other hand, the second scenario states that the first work station completes the production of all batches before the last work station begins production. Also, the optimal production run size and the optimal order quantities of raw materials are determined simultaneously by incorporating the cost of holding WIP inventory, production setup cost, and the classical EOQ model for ordering raw material. We have found in the first scenario that the optimal production run size and optimal order quantities of raw materials are influenced by the processing time of all work stations. On the contrary, in the second scenario the optimal production run size and optimal order quantities of raw materials are not affected by the processing time of any work station. In addition, by incorporating the cost of holding WIP inventory, production setup cost, and the classical EOQ model for ordering raw material a more realistic total production cost function is achieved which leads to a better batch size. This analysis can be extended by assuming that $P_1 > P_2 > \dots > P_m$ and this issue is currently being investigated.

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حجم الإنتاج الأمثل لنظام تصنيع متعدد المراحل *

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ملخص البحث. تم في هذا البحث تقديم نموذج رياضي لتحديد حجم الإنتاج الأمثل وكمية الطلب المثلى لمواد الخام لنظام تصنيع متعدد المراحل، حيث تم تحديد حجم الإنتاج الأمثل وكمية الطلب المثلى لمواد الخام بشكل متزامن مع الأخذ بالاعتبار تكاليف الاحتفاظ بالمخزون تحت التصنيع، وتكاليف الإعداد للتصنيع، ونموذج كمية الطلب الاقتصادية لمواد الخام. ولتحديد حجم الإنتاج الأمثل تم تحليل المخزون تحت التصنيع لحالتين في الحالة الأولى يفترض أن آخر محطة تشغيل تبدأ الإنتاج قبل انتهاء المحطة الأولى من إنتاج جميع دفع الإنتاج، أما الحالة الثانية تفترض أن المحطة الأولى تنهي إنتاج جميع الدفعات قبل بدء المحطة الأخيرة.

* لقد تم تقديم نسخة أولية من هذا البحث في المؤتمر الدولي عن إدارة سلسلة العرض، بكين، الصين، أغسطس ٢٠٠٢م.